Research Article

Open Access Journal of Physics & Mathematics

Received: 02-01-2025 **Accepted:** 07-01-2025 **Published:** 15-01-2025

Volume 1 | Issue 1

Maxwell-Lorentz Theory of Gravito-Rotational/-Magnetic Fields

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***Citation:** **Chandramohanan MR (2025) Maxwell-Lorentz Theory of Gravito-Rotational/-Magnetic Fields. Open Access Journal of Physics & Mathematics, Research Article 1: 1-47.**

Abstract

The article attempts at removing the deficiencies of the Newtonian theory and transforming it into a theory, equivalent to electro dynamics. The GTR is mostly concerned with a static theory of gravitation. On the other hand, the basic weakness of Newtonian theory is the static nature of the law of gravitation and absence of vector potential of AMPERE (1775–1836). In the following sections, we examine the non-relativistic theory and relativistic theory of motion and briefly describe the steps in order to remove the deficiencies of the Newtonian dynamics

Keywords and Notations: Local time interval (dt), proper time interval $(\mathrm{d\tau})$, gravitational field intensity $(\mathbf{E}_1)$, electrostatic field intensity $(\mathbf{E}_2)$, gravitational permittivity of free space (ε_1), electrical permittivity of free space (ε_2), gravitational permeability (μ_1), electrical permeability (μ_2), four-potentials of gravitational fields $(\phi_1, \mathbf{A}_1)$, four-potentials of electromagnetic fields $(\phi_2, \mathbf{A}_2)$, four-potentials of combinations of fields $(\phi_3, \mathbf{A}_3)$, flow density/current density for gravitational particles $(\rho_1 \mathcal{V} \text{ or } \mathbf{J}_1)$ current density for electrons $(\rho_2 \mathcal{V} \text{ or } \mathbf{J}_2)$. $\varepsilon_1 \mu_1 c^2 = 1$ and $\varepsilon_2 \mu_2 c^2 = 1$, Minkowski's proper momentum $m\frac{dr}{d\tau}$, classical Lagrangian (L) proper, Lagrangian $\mathrm{L_p} = \mathrm{L}\,(dt/d\tau)^2$, m^- stands for $(-m)$ and E^- stands for $(-E)$, v_p for phase velocity, v_g for group velocity, such that $v_p v_g = c^2$.

1. Introduction

1.1 A Survey of Pre-relativistic Theory of Gravitation

Newtonian dynamics, electrodynamics and general relativity are theories considered as different from each other, although there are some common features. In the former two, the inverse square law of Newton and Coulomb are similar. Thus, there are similarities on gravito-statics and electro-statics. Yet, the theories of gravito-dynamics and electro dynamics are at present dissimilar. Since gravitational signals/waves and electro-magnetic signals/waves propagate with approximately the same speed c in free-space, it is feasible to combine both the theories into a single theory. It may seem that there is no analogue of a magnetic field in the case of charge-less particles. The fluid motion near a source/sink reveals that a laminar motion is accompanied by a vortex motion, causing eddies and turbulence. Hence, there must be a rotational field or vortex field, similar to the magnetic field, in the case of a moving particle. In this chapter, we shall derive the Maxwell-Lorentz equations in Newtonian dynamics as well.

This will reduce the differences between the former two theories. The third theory of general relativity, has discordance with the former two because of the exclusion of vector potentials and the use of $\mathrm{R}_{ij} = 0$, R_{ij} being Ricci tensor [20] along with variational equation $\delta(ds) = 0$, where ds is the four metric of proper time. Although the vector potentials are excluded, the scalar potential ϕ_1 of the Newtonian theory enter the discussion in the Schwarzschild solution of the metric. This shows that the modified Newtonian dynamics, containing the four-potentials of gravitation and electro-magnetism, as well as the four metric will include all the three theories. It will be proved that even without the four metric, the modified Newtonian theory can explain such phenomena as:

- (i) perihelion motion of the planets
- (ii) gravitational red-shift
- (iii) bending of light rays passing nearer to the sun, etc., which were earlier thought to be inexplicable with Newtonian theory.

In his book, '***Philosophia Naturalis Principia Mathematica***', SIR ISAAC NEWTON (1642–1727) introduced the inverse square law of gravitation. This law played an important role in the development of mechanics. Newtonian mechanics assumed the existence of absolute space and inertial frame. This concept was severely criticized by LEIBNITZ (1646–1716) who argued that there is no philosophical need [1,3,5,22] for any conception of space, apart from the relations of matter and object. None of the high-minded metaphysics had led to any idea about how to develop a dynamical theory that might challenge the Newtonian theory, until the advent of electromagnetic theory. Before J.C. MAXWELL (1831–1879), it was supposed that all laws of physics are invariant under the Galilean transformations. Nevertheless, the electromagnetic theory is in apparent disagreement with the principle of Galilean relativity and Galilean transformation. To remove this apparent disagreement, H.A. LORENTZ (1853–1928) introduced a new transformation. The formulae of Lorentz transformation were discoveries [8] made by Lorentz when he was studying the equations of electricity and magnetism.

In a lecture to the Congress of Arts and Science at St. Louis, USA, on 24th September 1904, HENRY POINCARE (1854–1912) gave a generalized form of a new principle: 'the principle of relativity'. According to the principle of relativity, he said that the laws of physical phenomena must be the same for a fixed observer as for an observer who has a uniform motion of translation relative to him. The apparent failure of the Michelson-Morley experiment in 1887 to determine the velocity of earth relative to the ether without consideration of ether-drift [1,2,3,21] led EINSTEIN (1879–1955) to postulate the constancy of speed of light as an axiom in his 1905 paper 'On the Electrodynamics of Moving Bodies'[13]. The apparent failure of the Newtonian theory is its inability to explain

(i) the perihelion shift of the planet Mercury

(ii) the gravitational red-shift

(iii) deflection of light by the Sun

1.2 Relativistic Theory of Gravitation

Newton was aware that the inertial mass entering in his discussion of motion might not be precisely the same as the gravitational mass appearing in his law of gravitation. Later in 1889, Roland von Eötvös concluded from his experiments that the difference between the ratio of inertial mass and gravitational mass, $\frac{m_i}{m_g}$ for wood and platinum was less than 10^{-9} [21]. Einstein was impressed with this experimental conclusion [7]. It served him to postulate the principle of equivalence $m_i = m_g$ in his theory of gravitation. He formulated the relativistic theory of gravitation in his paper 'The Foundations of General Relativity' in 1916. The GTR is dependent on the Riemannian geometry and requires a study of metrics, affine connections and curvature of space-time [20,21].

Einstein used the principle of equivalence to derive his field equations. The field equations for empty space are $R_{\mu v} = 0$ [13,15,20,21]. Schwarzschild applied these field equations to derive the static metric:

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right)dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 \qquad \text{(i)}$$

By making the substitution $\rho = \frac{1}{2}\left[r - MG + (r^2 - 2MGr)^{\frac{1}{2}}\right]$ or $r = \rho\left(1 + \frac{MG}{2\rho}\right)^2$, the above Schwarzschild metric has the isotropic form

$$d\tau^2 = \frac{\left(1 - \frac{MG}{2\rho}\right)^2}{\left(1 + \frac{MG}{2\rho}\right)^2} dt^2 - \left(1 + \frac{MG}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\varphi^2) \qquad \text{(ii)}$$

Eddington and Robertson suggested another possible metric

$$d\tau^2 = \left(1 - \frac{2\alpha\, MG}{\rho} + \frac{2\beta M^2G^2}{\rho^2}\right)dt^2 - \left(1 + \frac{\gamma MG}{\rho} + \cdots\right)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\varphi^2) \qquad \text{(iii)}$$

where α, β, and γ are unknown dimensionless constants [21] comparing this with the isotropic form (ii), the expected values of α, β, and γ must be equal to 1, subject to experimental verification. Einstein suggested the following tests of general relativity [21].

(A) The gravitational red shift of spectral lines

(B) The deflection of light by the Sun

(C) The precession of the perihelia of the orbits of the inner planets

(D) The time delay of radar echoes passing the Sun

Confirmation of (A) is just equivalent to the principle of equivalence. This implies α=1. The statements (B) and (D) can test whether γ≈1 whereas (C) verifies that 2γ-β=1. Thus (A) to (D) can be verified, if it is found experimentally that α=β=γ=1. In the book [21] we read that these values are experimentally verified. It may be noted that by redefining dτ by means of

$$d\tau^2 = \mathrm{Exp}\left(\frac{-2\mathrm{MG}}{\mathrm{R}}\right)dt^2 - \mathrm{Exp}\left(\frac{2\mathrm{MG}}{\mathrm{R}}\right)d\mathrm{R}^2 - \mathrm{R}^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

and/or replacing R by r/ρ and if the exponential is expanded up to terms of the second order, we get the metric (iii) of Eddington and Robertson and then (i) and (ii) so that we finally get $\alpha = \beta = \gamma = 1$. Thus, a verification of the tests of general relativity suggested by Einstein and others depends on arbitrary metrics suggested by several writers [5,17,19,21].

From the whole analysis, it can be concluded that the concepts of Centre of mass frame, the proper time interval $d\tau$ and the theory of retarded/advanced potentials can remove the deficiencies of the classical Newtonian dynamics. Besides, the modified theory is sufficient to explain the tests (A) to (D) more elegantly.

2. Lorentz Concept of Time and Relativistic Concept of Time

2.1 Lorentz' Concept of Time and LT

H.A. Lorentz developed the Maxwellian electromagnetic theory, applicable to an electron in motion [10,11,12,15], in the 1890s. It is widely recognized within the physics community that the Lorentzian theory of electrodynamics (ether-based underlying preferred frame) is indeed in accord with all that has been observed. Lorentz theory is based on the concept of retarded/advanced time [8,9] whereas SRT is based on two postulates plus the 'retarded time' concept, at a later stage when motion of electrons in an electromagnetic field, is considered. This shows that the relativistic theory and the postulates of SRT are superfluous and, depending on a single concept, Lorentz' theory is superior. This fact is clear from Pauli's comment:- Since electron theory is in agreement with the SRT, the latter cannot produce results which are not already contained in the pre-relativistic Lorentz' electron theory [18].

It may be noted that Newtonian dynamics is based on the assumption of instantaneous propagation of interaction but Roamer's studies showed that the velocity of light is finite. Later it was found to be $= 2.998 \times 10^8$ mps. The potential due to a charge Q at a distant point $P(x, y, z)$ from it can be taken as $\phi = \frac{Q}{r}$ in suitable units. In classical physics, it is considered that the influence of the source charge Q at the field point P is transmitted instantaneously. However, from experiments, instantaneous interaction is impossible [11]; the maximum possible velocity of interaction is, at the speed c of light/graviton/neutrino. The solution of Poisson equation $\nabla^2\emptyset = -4\pi\rho$ is $\emptyset = \iiint (\rho/r)dV$ whereas that of $\nabla^2\emptyset - c^{-2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho$ is given by $\emptyset = \iiint ([\rho]/r)dV$ where $[\rho]$ is evaluated at the retarded time $t - \frac{r}{c}$ and r is the distance between the source and the field point. Lorentz [8,9] observed that, if any change takes place in one of the interacting charges, it will influence the other charges, only after a lapse of time $\frac{r}{c}$. If $P(x, y, z)$ is a fixed point of S-frame, then $d\left(t - \frac{r}{c}\right) = dt$. Hence the local time dt of the S-frame and the retarded time interval are equal. But if $P(x, y, z)$ is a variable point of S then the retarded time interval and the local time interval are not equal.

Hence the local time intervals of two frames, in apparent relative motion, also will not be equal. Hence two observers in relative motion, cannot have a common local time. Hence, Lorentz introduced two local times- t, t'

It is further assumed that $t = 0 = t'$ when the origins O′ and O coincided during the motion of the mass particle m. The coordinates (x, y, z, ct) will be true for an observer situated in the immediate vicinity of P but these will be corrected to $\left[x, y, z, c\left(t - \frac{r}{c}\right)\right]$ for an observer situated at O. Hence if (x, y, z, ct) are the coordinates of an event at P then $(x, y, z, ct - r)$ will be the coordinates of the same event relative to O. This shows that observers, who are spatially separated, even if in the same frame of reference, do not observe a given event simultaneously.

After duration of time t, according to the real observer at O and t' according to the hypothetical observer placed at the moving mass at O′, the distances of the mass m in motion will be determined as follows. The projected distance of m or O′ is $v_0 t$, the retarded distance is $v_0 t_r$ and the current/present distance of m and the hypothetical observer O′ at m is $v_0 t'$. Lorentz assumed that the retarded time t_r is given by [8,9,18].

$$\boldsymbol{t_r = t - \frac{1}{c}|r - v_0 t_r|} \qquad (2.1.1)$$

and then derived the formula

$$\emptyset = \frac{Q}{\sqrt{[(x - v_0 t)^2/(1 - e^2)] + y^2 + z^2}} \qquad (2.1.2)$$

for the potential at $P\ (x, y, z)$ due to the moving charge, where $e = \frac{v_0}{c}$. Let $x' = \gamma(x - v_0 t), y' = y, z' = z$, $\gamma = 1/\sqrt{1 - e^2}$.

Here is the beginning of LT in electro-magnetism [8]

In this context, we shall modify (2.1.1) by the equation

$$t_r = t - \frac{1}{c}(|\boldsymbol{r} - \boldsymbol{v}_0 t_r|) \propto t' - \frac{r'}{c}$$

i.e. the retarded time in the S′-frame is proportional to the estimate of t_r in the S-frame.

$$\therefore\ t_r = t - \frac{1}{c}\,|\boldsymbol{r} - \boldsymbol{v}_0 t_r| = \lambda \left|t' - \frac{r'}{c}\right| \qquad (2.1.3)$$

where λ may depend on v_0 and $\lambda \to 1$ as $v_0 \to 0$.

From equation (2.1.3) we have $(ct_r - ct)^2 = r^2 + v_0^2 t_r^2 - 2\boldsymbol{v}_0.\boldsymbol{r}\ tr$

$$\therefore t_r = \frac{\left(t - \frac{\boldsymbol{v}_0.\boldsymbol{r}}{c^2}\right)}{1 - e^2} - \left[\frac{\left(t - \frac{\boldsymbol{v}_0.\boldsymbol{r}}{c^2}\right)^2}{(1 - e^2)^2} - \frac{\left(t^2 - \frac{r^2}{c^2}\right)}{1 - e^2}\right]^{\frac{1}{2}} \qquad (2.1.4)$$

omitting plus sign by avoiding `advanced time'.
Comparing RHS of (2.1.3) and (2.1.4) we have

$$t' = \frac{t - \frac{\boldsymbol{v}_0.\boldsymbol{r}}{c^2}}{\sqrt{1 - c^2}}, \lambda = \frac{1}{\sqrt{1 - c^2}} \text{ and } \frac{r'}{c} = \sqrt{t'^2 - \left(t^2 - \frac{r^2}{c^2}\right)} \text{ or } t^2 - \frac{r^2}{c^2} = t'^2 - \frac{r'^2}{c^2}$$

Inverting the equations for x' and t' we get the full set of equations of LT: $x = \gamma(x' + v_0 t'), t = \gamma(t - \frac{v_0 x}{c^2})$ since $\boldsymbol{v}_0.\boldsymbol{r} = v_0 x$. By letting $\boldsymbol{v}_0$ along x-axis we further have $y = y', z = z'$. From these it follows that $c^2dt^2 - dx^2 = c^2dt'^2 - x'^2$ and

$$dtdx = \left(\frac{1 - e^2}{1 + e^2}\right) dt'dx'$$

$$\text{or} \qquad dt'dx' = \left(\frac{1 - e^2}{1 + e^2}\right) dtdx$$

according as we choose the unprimed or primed co-ordinates as the real co-ordinates and the other as fictitious. Hence both of these systems of coordinates cannot be equally inertial systems. This means that the relativistic claim that there is no preferred frame of reference, is logically false. Hence there must be a preferred frame, which can be shown to be the frame having proper time and the proper co-ordinate x_τ obtainable from the local space-time co-ordinates $(ct, x)/(ct', x')$. From LT we have $c\tau = ct\sqrt{1 - \frac{x^2}{c^2t^2}} = ct\sqrt{1 - \frac{v^2}{c^2}}$ where $v = \frac{x}{t}$. This is the formula for time-dilation. Also the formula of Fitzgerald-Larmar-Lorentz contraction hypothesis [12,13,15,17,18]. implies $L_0 = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}}$ or $L_0 ct = Lct$ where L_0 is the proper length and L is its apparent length in motion with relocate $\boldsymbol{v}$. Replacing L_0 by x_τ, L by x and $v = \frac{x}{t}$ imply

$$x_\tau = \frac{x}{\sqrt{1-\frac{x^2}{c^2t^2}}} = \frac{ctx}{\sqrt{c^2t^2-x^2}} \quad \text{and} \quad c\tau = \sqrt{c^2t^2-x^2}$$

The inverse transformation is

$$c^2t^2 = \frac{1}{2}c^2\tau^2\left[\sqrt{1+\frac{4x_\tau^2}{c^2\tau^2}}+1\right]$$

$$x^2 = \frac{1}{2}c^2\tau^2\left[\sqrt{1+\frac{4x_\tau^2}{c^2\tau^2}}-1\right]$$

This can be generalized by including $y = y_\tau, z = z_\tau$.

It is clear that the Lorentzian supposition of a preferred frame is valid, by choosing the non-linear transformation involving $(c\tau, x_\tau, y_\tau, z_\tau)$ as the preferred frame coordinates.

2.2 Relativistic Concept of Time & LT

We shall begin the discussion by examining the contradictory opinions of a former relativist H. Dingle and two adherents of STR.

1. Prof. Herbert Dingle in his 1972 book [4] 'Science at the Cross-roads' wrote: 'the question is left by the experimenters to the mathematical specialists, who either ignore it or shroud it in various obscurities'. He continues that 'obviously something must be logically and mathematically wrong with either the LT or the principle of relativity or both, and that to be logically and mathematically consistent, one would have to jettison either LT or the principle of relativity or both'.

2. In his book, 'Cranks, Quarks & the Cosmos' (**published by Oxford University Press, 1997**) the author Jeremy Bernstein points out that "I would insist that any proposal for a radically new theory in physics or science, should contain a clear explanation of why the precedent science worked. Einstein did this as the first page of the paper, 'on the electro-dynamics of moving bodies' illustrates perfectly".

3. In the paper 'Special Relativity Invalid?" by Vesselin Petkov, [http://groupkos.com/rnboyd/special-relativity-invalid1.html] the author attempts to convince the readers with a slim hope that, "what I will write below might be helpful to some of the people who have reservations about relativity". The author continues "As a rule those who criticize it, do not see the whole picture - they pick up only individual relativistic results. I know this even from personal experience - for twenty years, all letters/papers claiming to have 'finished' relativity, sent to two departments (where I have worked) have been regularly forwarded to me".

Since the above opinions are conflicting with each other, it is very essential to examine the time-concept of Special Relativity in details. The first paper [13] on relativity, viz. "***On the Electrodynamics of Moving Bodies***", published in 1905, contains a definition of simultaneity. In §1 of the paper, the author considers the following thought experiment. Let there be two observers at A and B in a stationary system having clocks of identical time reading.

Let a ray of light start from A at 'A time' t_A towards B, reflected at B at 'B time' t_B back to A, arriving at A time t'_A.

The two clocks synchronize if $t_B - t_A = t'_A - t_B$

The author further assumed that $2\,\mathrm{AB}/(t'_{\mathrm{A}} - t_{\mathrm{A}}) = c$, as the velocity of light in empty space. From the above two equations, it is clear that $t_{\mathrm{A}} = t_{\mathrm{B}} - \mathrm{AB}/c$, *i.e.* the author has used the concept of retarded time to explain 'simultaneity'. After this, the paper continues, "It is essential to have time defined by means of stationary clocks in the stationary system, and the time now defined being appropriate to the stationary system, we call it the time of "stationary system". In §1 of the paper, the author has used the idea of retarded time $t - r/c$ with c in the denominator. But in §2 of the paper, dealing with the "relativity of lengths and time", we come across the following postulates of relativity, *viz.*

> **Postulate 1**. The laws by which the states of physical systems undergo change are not affected, whether these changes of state are referred to one or the other of the two systems of coordinates in uniform translatory motion.

> **Postulate 2**. Any ray of light moves in the stationary system of coordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body. Hence, $\text{velocity} = \frac{\text{light path}}{\text{time interval}}$, where time interval is to be taken in the sense of definition in §1.

Again the author considers the clock ray emission and reflection experiment with a difference and then assumes

$$t_{\mathrm{B}} - t_{\mathrm{A}} = r_{\mathrm{AB}}/(c - v) \text{ and } t'_{\mathrm{A}} - t_{\mathrm{B}} = r_{\mathrm{AB}}/(c + v)$$

where A and B are the ends of a moving rod of speed v and r_{AB} is its length in the stationary system. The above equations give

$t_{\mathrm{A}} = t_{\mathrm{B}} - r_{\mathrm{AB}}/(c - v)$ and $t_{\mathrm{B}} = t'_{\mathrm{A}} - r_{\mathrm{AB}}/(c + v)$

Again we see that the author uses the idea of retarded time with different velocities $(c - v)$ and $(c + v)$ in the denominator, whereas the correct denominator is c, from Lorentzian time-concept. It is clear that the author wrongly uses the concept of retarded time. In §3, dealing with the derivation of Lorentz transformation the author again uses the concept of retarded time restricted to x-axis, where we read the equation

$$\frac{1}{2}\left\{\tau(0,0,0,t) + \tau\left[0,0,0,t + \frac{x'}{c - v} + \frac{x'}{c + v}\right]\right\} = \tau\left[x',0,0,t + \frac{x'}{c - v}\right]$$

in which τ is the time in the S' frame (in the place of t' of the Lorentzian theory) Here also, we see that $c + v$ is a velocity greater than the velocity of light. This is contrary to the content of the second postulate of relativity and contrary to the 'principle of retarded time/advanced time'.

From the above analysis, it is clear that there is no new concept of time, in the SRT, which contains a faulty use of retarded/advanced time. In this context we prove the following.

Proposition 1

The postulates of special relativity do not prove the truth of the linear Lorentz transformation; it leads to a non-linear transformation with y≠y' and z≠z'.

To derive the transformation between S frame and S′ frame referred in the previous discussion, let us take $z = 0 = z'$ for convenience. Let $z = x + iy$ be the complex variable. The second postulate leads us to the assumption that the circle $|z| = ct$ is transformed into the circle $|z'| = ct'$. The most general bilinear transformation is $z'/ct' = e^{i\phi_0}(z - v_0 t)/(ct - \bar{v}_0 z/c)$ where $\bar{v}_0$ is the complex conjugate of v_0. Hence, we may take

$$L'z' = z - v_0 t,\ L'ct' = ct - \bar{v}_0 z/c \qquad (2.2.1)$$

where L′ is constant depending on v_0. Inverting these equations we have

$$z(1 - v_0^2 c^2)/L' = z' + v_0 t' \qquad (2.2.2)$$

$$ct(1 - v_0^2/c^2)/L' = ct' + \bar{v}_0 z'/c$$

which can be written in the form

$$Lz = z' + v_0 t' \qquad (2.2.3)$$

$$Lct = ct' + \bar{v}_0 z'/c \qquad (2.2.4)$$

where $LL' = 1 - v_0^2/c^2$.

Thus L and L′ are complex conjugates with magnitude $\sqrt{1 - v_0^2/c^2}$.

Hence, we may take $L = c^{i\emptyset_0}\sqrt{1 - v_0^2/c^2}$ and $L' = c^{-i\emptyset_0}\sqrt{1 - v_0^2/c^2}$.

Clearly

$$\tan\emptyset_0 = \frac{y' v_0/c}{ct' + v_0 x'/c} = \frac{y\, v_0/c}{ct - v_0 x/c} \qquad (2.2.5)$$

This transformation is non-linear and in general $y \neq y'$. Similarly $z \neq z'$. Hence the relativistic claim 'since there is no motion of S′-frame along y and z axis, it follows that $y = y'$, $z = z'$' is logically false. The fault lies in the assumption "a sphere of light $r^2 = c^2t^2$ centred at O is transformed into the sphere of light $r'^2 = c^2t'^2$ centred at O′ ".

2.3 Review of the literature on STR

We shall review two of the books dealing with special relativity. Whit row treats with such problems as universal time, individual time, mathematical time, and from these arrive at relativistic time [16]. The author presents seven axioms, and after detailed and subtle discussions, arrives at Einstein's results: $t = \frac{1}{2}(t_1 + t_2)$

and $r = \frac{1}{2}c(t_2 - t_1)$. This is actually the weak form of the concept of retarded/advanced time, since the above equations can be re-written as $t = t_2 - \frac{r}{c}$ and $t_1 = t - \frac{r}{c}$. From earlier discussions, we know that these weaker forms, not containing the relative velocity $\bar{v}_0$, can explain only the clock-synchronization and simultaneity, but not the truth of the LT. Knowing this fact, Einstein attempted to use the retarded time/advanced time by means of the expressions like $t - \frac{r}{c-v_0}, t + \frac{r}{c+v_0}$. This is wrong and contrary to the constancy postulate of STR as well as the postulate of retarded/advanced time. Thus the book does not contain anything different from other books on STR.

Now we shall examine Bergson [16] who made a serious study of STR, pin pointing the weak logical basis. In the book, the author explains the unexpected result of Michelson-Morley Experiment, by using the Lorentz-Fitzgerald theory of contraction of length of a moving body in the direction of its motion, in relation to ether and as compared with its length at rest in the ether. He supposes that the rest length ℓ_0 in ether will have the apparent length $\ell_0\sqrt{1 - \frac{v_0^2}{c^2}}$ when moving with velocity v_0 relative to ether and as judged from the ether. He further declares that the time of the system dilates in the ratio $1{:}\sqrt{1 - \frac{v_0^2}{c^2}}$. This is equivalent to the statement $d\tau = \sqrt{1 - \frac{v_0^2}{c^2}}\, dt$ or $dt \approx \sqrt{1 + \frac{v_0^2}{c^2}}\, d\tau$ known as the formula for time-dilation.

From these bases, he deduces the LT and various other results found in STR. Since these concepts have no discordance with the concepts of retarded/advanced time, the conclusions of Bergson are valid. He gives a very elaborate critical survey of some of the consequences drawn by STR. He raises the question as to, what extent the Einstein 'times' are real times. He proclaims that we cannot speak about a reality reigning without introducing consciousness. He declares: when we want to know if we have to do with a real time or a fictitious time, we only have to ask if the object presented could or could not be observed, become conscious. He elaborately discusses the relation between observations of events and processes by observers as well in the moving system S′ as in the system S at rest, and comes to the conclusion that, by comparison of the 'times' of observers in different systems, there is only one 'time' that is 'real', the time that is experienced by the real observer. The other times are fictions. Bergson expresses it thus: 'we thus always come back to the same point. There is one real time and the others are fictitious. What is a real time if not a time experienced, or which could be? What is an unreal, auxiliary, fictitious time, if not one which

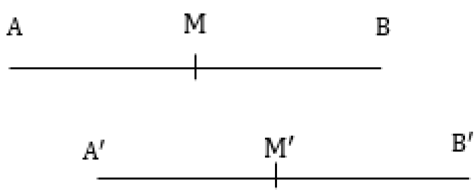

Figure 1

could not be effectively experienced by anything or anybody? 'The two observers in S and S' live exactly the same length of time and the two systems thus have the same real time. This is more closely discussed in connection with the train problem, which is the basis of Einstein's definition of simultaneity. He compares the observations of the observer on the embankment in the midpoint M between the points A and B where the hypothetical lightning strokes occur and the observer in the midpoint M' on the moving train between the points A' and B', where the lightning strokes occur with regard to the train. He comes to the result that one has to do with only one time.

What is simultaneous with regard to the embankment is also discordance with the postulate of retarded/advanced time, the conclusions obtained by Bergson simultaneous with regard to the train. Bergson comes here in flagrant opposition to Einstein's results. He explains it by pointing out that, we must suppose that the observations are really made by an observer - 'un-physician' -in the system. Only what this observer measures is real. But the observer can only be in one place. He is in M, and consequently cannot also be in M'. Bergson concludes that nothing has been really observed in M' because that would presuppose another observer in M', which is not the situation.

In a later discussion of the observations made by the observers in the two systems, one of which is resting on the earth, the other moving, he declares that the observer in the former system alone is real and the other observer a phantom. The exclusion of the privileged system of reference is the essence of STR. From the Lorentzian concept of time, we know that among the two local times t,t', only one is real, and the other is the correction or estimate to accommodate the relative motion of the single observer and the finite signal velocity of gravitational/electromagnetic signals. It is clear that Bergson's arguments are in agreement with the realistic version of relativity of H.A. Lorentz etc., but not in agreement with STR.

Bergson, in his criticism of Einstein's interpretation of the '**Simultaneity**' met with the rejoinder from Charles Nordmann [16] who argued that if Einstein's theory was really based on the demonstration of simultaneity, his theory would collapse and remarked that the real foundation of STR is to be found in Einstein's 1905 paper '**Electrodynamic**'. Failing to defend Bergson's criticisms, he virtually admitted the truth that the clock-synchronization, simultaneity etc. have no roles in the foundation of the theory of relativity. On the other hand, the realistic version of relativity based on the postulate of retarded/advanced time and the concept of proper time, has the correct logical foundations.

3. Gravito-statics

3.1 Classical Galilee - Newton Laws of Mechanics

Newton's laws of motion can treat the motion of macroscopic bodies such as vehicles, satellites, planets, etc. Newton's concepts are based on free space and inertial frames of reference. In all inertial frames of reference, an accelerated motion will have the same value for the acceleration and the laws of physics retain the same form.

Newton's laws are:

(i) A body continues to be in its state of rest or uniform linear motion, unless it is acted upon by an external force.

(ii) The rate of change of linear momentum of a body in motion is equal to the force acting on it.

(iii) If two bodies of masses m_1 and m_2 exert forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ on each other, then $\mathbf{F}_{12}+\mathbf{F}_{21}=\mathbf{0}$

These laws are invariant with respect to all Galilean transformations.

The fundamental problem of gravito-statics is to analyse the mutual action between pairs of a system of masses m_1 and m_2 ... m_r and their effect on a test-mass m and the trajectory of the test mass; by assuming that all the masses interact with each other and the interaction between any two masses is completely unaffected by the presence of the remaining masses. Hence, the principle of super position is applicable. Thus, the total gravitational force $\mathbf{F}$ on m is the sum of $\mathbf{F}_1, \mathbf{F}_2, \ldots$ due to $m_1, m_2, \ldots$ of the system.

$$\text{i.e. } \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots \quad (3.1.1)$$

The various components $\mathbf{F}_1, \mathbf{F}_2, \ldots$ can be found by using Newton's law of gravitation (Coulomb's law for charges) which had been found true by experiments.

Newton's law of gravitation states that the static gravitational attractive force between two mass particles of masses m_1, m_2, situated at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is

$$\mathbf{F}_{12} = \frac{G m_1 m_2 \boldsymbol{e}_{12}}{r_{12}^2} \quad [\text{N}] \quad (3.1.2)$$

and

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = \frac{G m_1 m_2 \boldsymbol{e}_{21}}{r_{21}^1} \quad [\text{N}] \quad (3.1.3)$$

where $\mathbf{F}_{12}$ is directed from m_2 to m_1, G is the gravitational constant,

ε_1 is the gravitational permittivity, $\boldsymbol{e}_{12} = \frac{\boldsymbol{r}_{12}}{r_{12}}$, $G = \frac{1}{4\pi\varepsilon_1}$ and $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

The value of $G = \frac{1}{4\pi\varepsilon_1} = 6.673 \times 10^{-11} [\text{N} \cdot \text{m}/\text{Kg}^2]$.

By replacing m_1 and m_2 by unlike charges Q_1 and Q_2 and ε_1 by ε_2 we get the Coulomb's law of electro statics.

3.1.1 Gravitational Field Intensity ($\boldsymbol{E}_1$)

The force of attraction due to a particle of mass m_1 situated at $Q(\boldsymbol{r}')$ on a test particle of unit mass at $P(\boldsymbol{r})$ is given by

$$\boldsymbol{E}_1 = \frac{m_1(-\hat{R})}{4\pi\varepsilon_1 R^2} \quad [\text{N}/\text{Kg}] \quad (3.1.4)$$

where $\mathbf{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $\hat{R}$ is a unit vector along $\mathbf{R}$. It is called the gravitational field intensity of m_1 at $P(\boldsymbol{r})$. The vector field defined by $\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1$ is called the gravitational flux density. Now by applying the principle of super position we rewrite (3.1.1)

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots = \frac{1}{4\pi\varepsilon_1}\left[\frac{mm_1}{R_1^2}\widehat{R}_1 + \frac{mm_2}{R_2^2}\widehat{R}_2 + \cdots\right]$$

$$= \frac{m}{4\pi\varepsilon_1}\left(\frac{m_1\widehat{R}_1}{R_1^2} + \frac{m_2\widehat{R}_2}{R_2^2} + \ldots\right) = m\mathbf{E} \qquad (3.1.5)$$

where

$$\mathbf{E}\,(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_1}\sum_i \frac{m_i\widehat{\boldsymbol{R}}_i}{R_i^2} \qquad (3.1.6)$$

is called the gravitational field intensity of the system of masses m_1, m_2,... and $\mathbf{R}_i = \boldsymbol{r} - \boldsymbol{r}_i'$, $\boldsymbol{r}_i'$ being the location of m_i. For a continuous mass distribution of volume V', the equation (3.1.6) can be modified to

$$\mathbf{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_1}\int_{V'}\frac{\widehat{\mathbf{R}}dm}{R^2} = \frac{1}{4\pi\varepsilon_1}\int_{V'}\rho(\boldsymbol{r}')\frac{\widehat{\mathbf{R}}dV'}{R^2} \qquad (3.1.7)$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $\widehat{\mathbf{R}}$ is the unit vector along $\mathbf{R}$.

3.1.2 Energy Stored in a Gravitational Field

The potential energy per unit mass or potential due to a mass m_1 at a field point $P(\boldsymbol{r})$ is defined by

$$\phi_1 = \frac{m_1}{4\pi\varepsilon_1 R} \quad [J/Kg] \text{ or } \left[N\cdot\frac{m}{Kg}\right] \qquad (3.1.8)$$

If a mass m_2 is placed at $P(\boldsymbol{r})$ in the field of m_1 then the potential energy at m_2 in the field of m_1 is

$$W = \frac{m_1 m_2}{4\pi\varepsilon_1 R} = m_2\,\phi_1 \quad [N\cdot m] \text{ or } [J] \qquad (3.1.9)$$

For a system of particles $m_1, m_2, \ldots$ we define the potential energy of the field as:

$$W = \frac{1}{2}[m_1\phi'_1 + m_2\phi'_2 + \cdots] \qquad (3.1.10)$$

where

$$\phi'_1 = \sum_{i\neq 1}\frac{m_i}{4\pi\varepsilon_1 R_{i1}},\ \phi'_2 = \sum_{i\neq 2}\frac{m_i}{4\pi\varepsilon_1 R_{i2}} \text{ etc.}$$

For a system of continuous distribution of masses, (3.1.10) becomes

$$\mathrm{W} = \frac{1}{2}\int \phi'_1 \rho_1 d\mathrm{V}_1 + \frac{1}{2}\int \phi'_2 \rho_2 d\mathrm{V}_2 + \cdots \qquad (3.1.11)$$

where the region of integration does not contain the volume of the source so as to cut off singularity.

3.1.3 Removable singularity of potential function-centre of mass

First, we will resolve the singularity of the potential function representing potential energy per unit mass/ unit charge. The potential due to a spherical point mass M at a distance r from its centre of mass is given by $\phi_1 = \mathrm{MG}/r$. Consider the potential at $\boldsymbol{r}$ due to the presence of two masses m_1 and m_2 as given by

$$\phi_{12} = \frac{m_1}{|\boldsymbol{r} - \boldsymbol{r}_1|} + \frac{m_2}{|\boldsymbol{r} - \boldsymbol{r}_2|} \qquad (3.1.12)$$

where we took $\mathrm{G} = 1, \boldsymbol{r}_1$ and $\boldsymbol{r}_2$, the locations of $m_1$and m_2. For convenience, we may take $y = 0, z = 0, y_1 = 0, z_1 = 0$ and $x_1 = a, x_2 = b$

$$y_1 = 0, z_2 = 0$$

$$\therefore \phi_{12} = \frac{m_1}{x - a} + \frac{m_2}{x - b} \qquad (3.1.13)$$

Let $g = \frac{m_1 b + m_2 a}{m_1 + m_2}$ and $g' = \frac{m_1 a + m_2 b}{m_1 + m_2}$

$$\therefore \phi_{12} = \frac{(m_1 + m_2)x - (m_1 b + m_2 a)}{(x - a)(x - b)}$$

$$\therefore \phi_{12} = \frac{(m_1 + m_2)(x - g)}{(x - g)(x - g') - \mathrm{A}^2} \qquad (3.1.14)$$

where $\mathrm{A}^2 = \frac{m_1 m_2 (b - a)^2}{(m_1 + m_2)^2}$

Equation (3.1.14) shows that the potential due to two masses will vanish at the centre of mass $x = g$. The conclusions are that, (i) for any system of particles, the origin of potential (zero potential) should be at the centre of mass; (ii) for a single point mass M the potential at its centre of mass must be zero. This assertion demands that the potential function for a single mass M, defined earlier must be chosen in the form $\phi_1 = \frac{[\mathrm{M}]\mathrm{G}}{|\boldsymbol{r} - \boldsymbol{r}_1|}$ when $|\boldsymbol{r} - \boldsymbol{r}_1| = d_1 \leq$ radius of the spherical mass, where [M] is the effective mass within the spherical region, *i.e.* within $|\boldsymbol{r} - \boldsymbol{r}_1| = d_1$. Outside this region we may take $\phi_1 = \frac{\mathrm{MG}}{|\boldsymbol{r} - \boldsymbol{r}_1|}$ as usual. Similarly for a charge

q, we define $\Phi_2 = \left(\frac{q}{m}\right)\frac{[m]}{4\pi\varepsilon_2|\boldsymbol{r}-\boldsymbol{r}_1|}$, when $|\boldsymbol{r}-\boldsymbol{r}_1| = d_1 \leq$ radius of the spherical charge. Outside this region we may take $\Phi_2 = \frac{q}{4\pi\varepsilon_2|\boldsymbol{r}-\boldsymbol{r}_1|}$, $|\boldsymbol{r}-\boldsymbol{r}_1| >$ radius of the electron, m being the mass of the electron. Clearly both Φ_1 and Φ_2 tend to zero when the radius of $[\mathrm{M}]$ and $[m]$ tend to zero.

3.1.4 Resolution of Singularity of Field Energy

In this section, we attempt to remove the singularity of field energy due to an electron/mass particle, by using Bohr's principles. In electromagnetism, we have the equation that the electromagnetic field energy of a point charge of radius a is given by [8] $\mathrm{U}_{elec} = \frac{q^2}{8\pi\varepsilon_0 a}$. Similarly, the energy of a charged sphere of radius a is also given by $\mathrm{U}_{elec} = \frac{\mathrm{Q}^2}{8\pi\varepsilon_0 a}$. Therefore, total energy in the field and the energy within the charge are each equal to $\frac{\mathrm{Q}^2}{8\,\pi\,\varepsilon \mathrm{A}}$. When a tends to zero, the conclusion is that there is an infinite amount of energy in the field of a point charge or charged sphere. Applying Bohr's theory of hydrogen atom, we can remove the apparent singularity stated above. We can re-write the above formula in the form

$$\mathrm{U}_{elec} = \frac{\mathrm{Q}^2}{8\pi\varepsilon_0 a} = \frac{\mathrm{Q}^2}{8\pi\varepsilon_0}\sum_1^\infty \frac{1}{n(n+1)a} \quad \text{since} \quad \sum_1^\infty \frac{1}{n(n+1)} = 1$$

$$= \sum_1^\infty \frac{\mathrm{Q}^2}{8\pi\varepsilon_0 r_n} = \sum_1^\infty \mathrm{E}_n$$

where $\mathrm{E}_n = \frac{\mathrm{Q}^2}{8\pi\varepsilon_2 r_n}$, and ε_2 is our notation for ε_0 for a charge E_n is the energy level at a radial distance $r_n = n\,(n+1)\,a$, $a = r_0$ or an integral multiple of r_0 and r_0 is the Bohr radius. But Bohr uses $r_n = n^2 r_0$ instead of $n(n+1)r_0$. Thus, the field energy at r_n from a charge is E_n and the total energy in the field is $\mathrm{U} = \frac{\mathrm{Q}^2}{8\pi\varepsilon_2 a}$. We can extend this result to mass particle of radius a . The field energy due to a mass particle of mass M may be taken $\mathrm{U} = \frac{\mathrm{M}^2}{8\pi\varepsilon_1 a}$. When $a \to 0$. We replace M by [M] and Q by $\left(\frac{\mathrm{Q}}{\mathrm{M}}\right)$ [M].

Similarly, we can write the corresponding equations in electro-statics.Thus, there is perfect reciprocity between static gravitational fields and electro-static fields. This similarity is not found when we compare electro dynamics with the existing gravitational-dynamics. Therefore, we derive those equations, which have at present no place in gravito-dynamics, but occurring in electro dynamics.

4. Analysis of Maxwell-Lorentz Theory

In this section, we shall derive Gauss' Laws, non-relativistic Dirac's equation, Faraday's Law, Ampere's Law, Lorentz Force Law and Gravitational Wave Equations.

4.1 Gauss' Laws

Let M be the mass of a particle moving with velocity $\boldsymbol{v}$; let it be at $Q\ (x', y', z')$ at some instant of time. Consider a spherical cap of central angle 2θ, cut off from a sphere centred at M, by a plane at a distance X, so that $\boldsymbol{v}$ is normal to the base of the cap and the radius of the circular rim of the cap is given by

$$r_x{}^2 = Y^2 + Z^2 \text{ and } |\mathbf{R}|^2 = X^2 + Y^2 + Z^2, X = x - x', Y = y - y', Z = z - z'$$

Since the diameter through $P(x, y, z)$ of the rim C, subtends 2θ at the centre Q of the sphere, the area of the cap is

$$A_\theta = 2\pi r^2(1 - \cos\theta) \qquad (4.1.1)$$

and the specific area of the cap is

$$\frac{A_\theta}{A} = \frac{1}{2}(1 - \cos\theta) \qquad (4.1.2)$$

In contrast to equation (3.1.4), but in accordance with electrodynamics, we introduce the

Definition

At each point on the cap, a vector of uniform magnitude in the direction of the radial line, emanating from the particle at Q will be defined; the flux ψ of this vector field $\boldsymbol{E}_1$ (gravitational field intensity) over the cap is assumed proportional to (i) the mass M of the particle and (ii) the specific area

$$i.e.\ \varepsilon_1\ \psi_1 = \iint \varepsilon_1 \mathbf{E}_1 \cdot \boldsymbol{n} dS = \frac{1}{2} M\ (1 - \cos\theta) \qquad (4.1.3)$$

where ε_1, the permittivity of space, is the proportionality factor.

$$\therefore \iint \mathbf{D}_1 \cdot\ \boldsymbol{n} dS = \frac{1}{2} M\ (1 - \cos\theta) \qquad (4.1.4)$$

where the integration is performed on the cap and $\mathbf{D}_1 = \varepsilon_1\ \mathbf{E}_1$ is defined as the gravitational flux density.

By letting $\theta = \pi$ in (4.1.4), the flux of $\boldsymbol{D}_1$ over the sphere is given by

$$\oiint \mathbf{D}_1 . \boldsymbol{n} dS = M \qquad (4.1.5)$$

$$\text{But} \iiint \rho_1 dV = M \qquad (4.1.6)$$

Hence $\text{Div}\ \mathbf{D}_1 = \rho_1$

$$\text{or Div } \mathbf{E}_1 = \frac{\rho_1}{\varepsilon_1} \tag{4.1.7}$$

Changing $\mathbf{E}_1$ to $\mathbf{E}_1^- = -\mathbf{E}_1$ and ρ_1 to ρ_1^-, we have

$$\text{Div } \mathbf{E}_1^- = \frac{\rho_1^-}{\varepsilon_1} \tag{4.1.7'}$$

The last two equations are Gauss' Law for repulsive/attractive gravitational fields.

In the case of an oblique surface, the surface is divided into elemental areas, the normal components of $\mathbf{E}_1$ for each elemental surface, multiplied by the area of the element and summed for the whole surface. *More generally a vector field can be replaced by a contracted tensor field.* It can be verified readily, that the Newtonian choice

$$\mathbf{D}_1 = \frac{M\hat{R}}{4\pi R^2} = \varepsilon_1 \mathbf{E}_1 \tag{4.1.8}$$

$$\mathbf{E}_1 = -\nabla\phi_1 \quad \text{and} \quad \phi_1 = \frac{M}{4\pi\varepsilon_1 R} \tag{4.1.9}$$

satisfy the requirements of the definition, when the particle is not in motion. But when the test particle m is in motion, these have to be modified subject to (i) the equation of continuity (gauge condition) and (ii) the variation of mass with velocity.

4.2 Ampere's Circuital Law

Differentiating (4.1.4) w.r.t. time we have,

$$\frac{d}{dt}\iint \mathbf{D}_1 \cdot \boldsymbol{n} dS = \frac{1}{2} M \sin\theta \frac{d\theta}{dt} + \frac{1}{2}(1-\cos\theta)\frac{dM}{dt} \tag{4.2.1}$$

The second term on the RHS, containing $\frac{dM}{dt}$ is negligibly small, since it depends on c^{-2}. Discarding that term

$$\text{RHS} = \frac{1}{2} M \sin\theta \frac{d\theta}{dt}$$

$$= \frac{1}{2} M \frac{r_x}{R} \frac{d}{dt}\left(\tan^{-1}\frac{r_x}{X}\right)$$

where $X = x - x', r_x = \sqrt{(y-y')^2 + (z-z')^2}$

$$= \frac{1}{2} M \frac{r_x(X\dot{r}_x - r_x \dot{X})}{R^3}$$

$$= \frac{1}{2} M \; r_x \frac{|\mathbf{R} \times \dot{\mathbf{R}}|}{R^3}$$

$$= 2\pi\, r_x \left|\frac{M\mathbf{R}}{4\pi R^3} \times \dot{\mathbf{R}}\right|$$

$$= 2\pi\, r_x |\mathbf{D}_1 \times\ \dot{\mathbf{R}}|$$

$$= 2\pi\, r_x |\varepsilon_1 \mathbf{E}_1 \times\ (-\mathbf{v})|$$

$$= 2\pi\, r_x \varepsilon_1 |\boldsymbol{v}\ \times\ \mathbf{E}_1|$$

$$= 2\pi\varepsilon_1\, r_x |c^2\ \mathbf{B}_1|$$

$$= 2\ \pi\ \varepsilon_1 c^2\ \mu_1\ \ r_x |\mathbf{H}_1|$$

$$= 2\ \pi\ r_x |\mathbf{H}_1|$$

$$= \oint \mathbf{H}_1 \cdot d\boldsymbol{r}$$

$$= \iint (\nabla \times \mathbf{H}_1) \cdot \boldsymbol{n} dS \qquad (4.2.2)$$

where we used

$$c^2 \mathbf{B}_1 = \boldsymbol{v}\ \times\ \mathbf{E}_1 \qquad (4.2.3)$$

$$\varepsilon_1 \mu_1 c^2\ =\ 1 \qquad (4.2.4)$$

and

$$\mathbf{B}_1 =\ \mu_1\ \mathbf{H}_1 \qquad (4.2.5)$$

and the integration is performed along the rim C of the cap for the circulation and the spherical cap S for the flux. Besides S can be extended to the whole surface of the conical frustum, since the components of $\mathbf{D}_1/\mathbf{H}_1$ normal to the conical part of the surface is zero. Hence, (4.2.2) holds for the complete surface Σ of the conical frustum. By using the divergence theorem,

$$\text{LHS of (4.2.1)}\ = \frac{d}{dt} \iiint (\text{Div } \mathbf{D}_1)\ dV$$

$$= \frac{d}{dt} \iiint \rho_1\ dV$$

$$= \iiint \left[\frac{\partial \rho_1}{\partial t} + \text{Div } (\rho_1 \boldsymbol{v})\right]\ dV$$

$$= \iiint \left[\frac{\partial}{\partial t}\ (\text{Div } \mathbf{D}_1)\ +\ \text{Div } \mathbf{J}_1\right]\ dV$$

$$= \iiint \text{Div } (\mathbf{J}_1 + \frac{\partial \mathbf{D}_1}{\partial t})\ dV$$

$$= \iint \left(\mathbf{J}_1 + \frac{\partial \mathbf{D}_1}{\partial t}\right) \cdot \boldsymbol{n} dS \qquad (4.2.6)$$

where the region of integration is the complete surface Σ of the conical frustum and V is its volume. From (4.2.2) and (4.2.6) we have

$$\nabla \times \mathbf{H}_1 = \mathbf{J}_1 + \frac{\partial \mathbf{D}_1}{\partial t} \qquad (4.2.7)$$

This is Ampere's circuital law.

4.2.1 Modification of Static Potentials by Retarded Potentials

From equation (4.1.9) we have

$$\phi_1 = \frac{\mathrm{M}}{4\pi\varepsilon_1 \mathrm{R}}$$

for static potentials.

In accordance with the theory of retarded potentials [8,9] since M is moving with velocity v, we replace $\mathrm{R} = |\boldsymbol{r} - \boldsymbol{v}\, t_r|$ of Section 2.1 by the present distance $r' = \frac{\mathrm{R} - \frac{\mathbf{R}\cdot v}{c}}{\left|1 - \frac{v^2}{c^2}\right|^{\frac{1}{2}}}$ in the notations of (2.1.2) and (2.1.3).

Thus, we take $\mathrm{M}\sqrt{1 - e^2} = \mathrm{M}_0$

$$\phi_1 = \frac{\mathrm{M}}{4\pi\varepsilon_1 r'} = \frac{\mathrm{M}_0}{4\pi\varepsilon_1 \left(R - \frac{\boldsymbol{v}\cdot\mathbf{R}}{c}\right)} \qquad (4.2.8)$$

$$\mathbf{A}_1 = \frac{\phi_1 \boldsymbol{v}}{c^2} = \frac{\mu_1 \mathrm{M}_0 \boldsymbol{v}}{4\pi\left(\mathrm{R} - \frac{\boldsymbol{v}\cdot\mathbf{R}}{c}\right)} \qquad (4.2.9)$$

We have the constitutive relations $\mathbf{D}_1 = \epsilon_1 \mathbf{E}_1$ and $\mathbf{B}_1 = \mu \mathbf{H}_1$ for gravitational fields. We shall modify the vectors **D** and **B** by using contracted tensor fields. The metric in tensor calculus [15] is given by $(ds)^2 = g_{ij} dx^i dx^j = dx_j dx^j$ where g_{ij} is a $(0,2)$ tensor and dx_j, dx^j are the covariant and contra-variant components of the same vector. Let us introduce, for the gravitational field, two reciprocal/conjugate tensors ϵ_{ij} and ϵ^{ij} such that $\epsilon^{i\alpha}\epsilon_{\alpha j} = \epsilon^2 \delta^i_j$. $D_i = \epsilon_{i\alpha} E^\alpha$ and $D^i = \epsilon^{i\alpha} E_\alpha$ as the covariant and contra-variant components of D-field.

Now $\epsilon_{ij} E^i E^j = (\epsilon_{ij} E^i) E^j = E_j E^j = |\mathbf{E}|^2$ and $|\mathbf{D}|^2 = D_i D^i = (\epsilon_{i\alpha} E^\alpha)(\epsilon^{i\beta} E_\beta)$

$$= \epsilon^2 \delta_\alpha^\beta E^\alpha E_\beta = \epsilon^2 E^\alpha E_\alpha = \epsilon^2 |\mathbf{E}|^2$$

so that $|\mathbf{D}| = \epsilon|\mathbf{E}|$ holds.

Similarly we introduce two conjugate tensors μ_{ij} and μ^{ij} such that $\mu^{i\alpha}\mu_{\alpha j} = \mu^2 \delta_j^i$ and define $B_i = \mu_{i\alpha} H^\alpha$, $B^i = \mu^{ij} H_j$ as the covariant and contra-variant components of $\boldsymbol{B}$ where ϵ, μ stands for ϵ_1, μ_1 in the gravitational case and ϵ_2, μ_2 for the EM-fields, such that $\epsilon\mu c^2 = 1$ where c is the maximum signal velocity of the gravito-rotational/ EM fields. Hence

$$B_i B^i = (\mu_{i\alpha} H^\alpha)(\mu^{i\beta} H_\beta) = \mu^2 \delta_\alpha^\beta H^\alpha H_\beta = \mu^2 H^\alpha H_\alpha = \mu^2 |\boldsymbol{H}|^2$$

so that $|\boldsymbol{B}| = \mu|\boldsymbol{H}|$ holds.

Hence our tensors defined above have the properties that

(i) ϵ_{ij} and ϵ^{ij} are conjugate tensors dependent on ϵ_1/ϵ_2 and $\epsilon_{ij}\epsilon^{ij} = \epsilon^2$

(ii) μ_{ij} and μ^{ij} are conjugate tensors dependent on μ_1/μ_2 and $\mu_{ij}\mu^{ij} = \mu^2$

(iii) ϵ's and μ's are dependent on the dielectric parameter ϵ_1 and the gravitational susceptibility μ_1, for the gravitational fields and ϵ_2, μ_2 for EM fields, such that $\epsilon\mu c^2 = 1$ for both fields. Hence $dE^2 - d\boldsymbol{H}^2$ is a metric for the gravito-rotational/EM fields $dE^2 - dH^2 = \epsilon_{ij} dE^i dE^j - \mu_{ij} dH^i dH^j$ with parameters $(\epsilon_1, \mu_1)/(\epsilon_2, \mu_2)$.

We shall verify that these choices will satisfy $\mathbf{B_1} = \frac{v \times \mathbf{E_1}}{c^2}$ and Div $\mathbf{B}_1 = 0$

$$\text{Curl } \mathbf{A}_1 = \frac{\mu_1 \mathrm{M}_0}{4\pi} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{v_x}{\mathrm{R} - \frac{\boldsymbol{v} \cdot \mathrm{R}}{c}} & \frac{v_y}{\mathrm{R} - \frac{\boldsymbol{v} \cdot \mathrm{R}}{c}} & \frac{v_z}{\mathrm{R} - \frac{\boldsymbol{v} \cdot \mathbf{R}}{c}} \end{vmatrix}$$

$$= \frac{-\mu_1 \mathrm{M}_0}{4\pi} \sum \left[\frac{v_z \left(\frac{y - y'}{\mathrm{R}} - \frac{v_y}{c} \right) - v_y \left(\frac{z - z'}{\mathrm{R}} - \frac{v_z}{c} \right)}{\left(\mathrm{R} - \frac{\boldsymbol{v} \cdot \mathbf{R}}{c} \right)^2} \right] \boldsymbol{i}$$

$$= \frac{-\mu_1 \mathrm{M}_0}{4\pi} \sum \left[\frac{v_z (y - y') - v_y (z - z')}{\mathrm{R}\left(R - \frac{\boldsymbol{v} \cdot \mathbf{R}}{c} \right)^2} \right] \boldsymbol{i}$$

$$= \frac{\mu_{1M_0 v}}{4\pi} \times \frac{\mathbf{R}}{\mathrm{R}\left(-\frac{\boldsymbol{v} \cdot \mathbf{R}}{c}\right)^2}$$

$$= \frac{\boldsymbol{v}}{c^2} \times \frac{\mathrm{M_0}\widehat{\mathrm{R}}}{4\pi\varepsilon_1\left(\mathrm{R} - \frac{\boldsymbol{v} \cdot \mathbf{R}}{c}\right)^2}$$

$$\therefore \text{Curl } \mathbf{A}_1 = \frac{\boldsymbol{v} \times \mathbf{E}_1}{c^2} = \mathbf{B}_1 \qquad (4.2.10)$$

by using equation (4.2.3) where

$$\mathbf{D}_1 = \frac{\mathrm{M_0}\widehat{\mathrm{R}}}{4\pi\left(\mathrm{R} - \frac{\boldsymbol{v} \cdot \mathbf{R}}{c}\right)^2} = \varepsilon_1 \mathbf{E}_1 \qquad (4.2.11)$$

This is the generalization for equation (4.1.8) when the mass particle is in motion. Clearly

$$\text{Div } \mathbf{B}_1 = \text{Div Curl } \mathbf{A}_1 \equiv 0 \qquad (4.2.12)$$

Equations (4.2.8) to (4.2.11) contain the generalization of the earlier equations (4.1.8), (4.1.9), and (4.2.3). Thus, instead of the single scalar potential ϕ_1, we have the four-potentials $(\phi_1, \mathbf{A}_1)$. The foregoing analysis shows that the gravitational fields of a moving particle of mass M consists of a scalar potential ϕ_1 and a vector potential $\mathbf{A}_1$, known as Ampere's vector potential. These are exactly similar to the four-potentials of electro-magnetic theory. In the above analysis, if we assume M to be at rest, and consider a test particle of mass m moving with velocity $\boldsymbol{v}$ in the field of M, then the fields due to M at m can be obtained from the above equations by replacing $\boldsymbol{v}$ by $(-\boldsymbol{v})$. Since a charged particle has both mass and charge, it has gravitational as well as electromagnetic fields. Hence any external four potentials will be the sum or super position of gravitational and electromagnetic potentials in the form $\phi_3 = \phi_1 \pm i\frac{q}{m}\phi_2$ and $\mathbf{A}_3 = \mathbf{A}_1 \pm i\frac{q}{m}\mathbf{A}_2$ where $\frac{q}{m}(\phi_2, \mathbf{A}_2)$ is the four potentials due to electro-magnetism. This statement is based on the following considerations; ϕ_1 being energy/unit mass has dimension of c^2 and ϕ_2 being energy per unit charge has dimension of $\frac{mc^2}{q}$ so that ϕ_1 and $\frac{q}{m}\phi_2$ have dimension of c^2 and we write $\phi_3 = \phi_1 \pm i\frac{q}{m}\phi_2$. Similarly, $\mathbf{A}_1 = \frac{\phi_1 v_1}{c^2}$ has dimension of velocity and $q\mathbf{A}_2 = \frac{q\phi_2 v_2}{c^2}$ has the dimension of energy times $\frac{v_2}{c^2}$ i.e. $q\mathbf{A}_2$ has dimension of momentum so that $\frac{q}{m}\mathbf{A}_2$ has dimension of velocity. Hence, we may write $\mathbf{A}_3 = \mathbf{A}_1 \pm i\frac{q}{m}\mathbf{A}_2$. Further there is no experimental confirmation of any interaction between gravitation and electro magnetism[6]. In this general case, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the average velocities of test mass/test charge, relative to the centre of mass/centre of charge. Since the Hamiltonian for the motion of a particle of mass m is $\mathrm{H}_1 = mc^2 + m\phi_1$, we can re-write it as $\mathrm{H}_1 = \mathrm{E}_1 = \left(m + m\frac{\phi_1}{c^2}\right)c^2 = m^*c^2$ where $m^* = m + m\frac{\phi_1}{c^2}$. This shows that mass consists at least of an inertial part and a gravitational part. Hence the momentum of a charged particle in general can be taken as $\boldsymbol{p} = m\boldsymbol{v} + m\mathbf{A}_1 \pm iq\,\mathbf{A}_2$ and the mass energy relation may be taken as E=m^*c^2, where $m^* = m + m\frac{\phi_1}{c^2} \pm \frac{iq\phi_2}{c^2}$.

That is mass/energy consists of

(i) an inertial part/mechanical part,

(ii) a gravitational part and

(iii) an electro-magnetic part/quantum mechanical part.

Newton never treated inertial mass and gravitational mass as equal which is in conformity with the above discussion, but contrary to the principle of the equivalence of GTR. We have derived the field equations for a single particle. When we consider fluid motion, an elemental volume dV of a fluid in motion will have a large number of particles and the space between particles may not be free. Hence the permittivity ε_1 and permeability μ_1 will have to be modified. Moreover, interactions and frictional forces make the modification complex. Applications to fluid dynamics will be continued in Section 5.

4.3 Mass-Velocity Relations

For a free particle, the Lagrangian L^* and the Hamiltonian H represent the energy, one in the moving frame of the particle, and the other in the laboratory frame of the observer. Hence it is possible that both can be expressed in the form mc^2 where m is a function of the velocity $\boldsymbol{v}$ of the particle. This is done by using the defining equations in classical dynamics, namely,

$$\left(\frac{\partial}{\partial \boldsymbol{v}}\right)\mathrm{L}^* = \text{momentum} = m\boldsymbol{v} \qquad (4.3.1)$$

$$\text{and } \left(\frac{\partial}{\partial p}\right)\mathrm{H} = \boldsymbol{v} \qquad (4.3.2)$$

Thus, by letting $\mathrm{L}^* = f(\boldsymbol{v})\, m_0c^2 = mc^2$ in (4.3.1) and taking $\boldsymbol{v}$ along x-axis, we have

$$f'(v)\, m_0c^2 = c^2\frac{dm}{dv} = mv$$

$$\therefore \frac{dm}{m} = \frac{v dv}{c^2} \cdot \text{ Integrating}$$

$$m = m_0 \mathrm{Exp}\left(\frac{v^2}{2c^2}\right) \qquad (4.3.3)$$

By defining $H = \boldsymbol{v} \cdot \left(\frac{\partial}{\partial v}\right) L^* - L^*$

it is seen that (4.3.2) is satisfied. It can be similarly shown that the Lorentzian mass given by[10, 12]

$$m = \frac{m_0}{\sqrt{\left|1 - \frac{v^2}{c^2}\right|}} \qquad (4.3.4)$$

can be obtained from (4.3.2) and the assumption. $mc^2 = g(v)\, m_0 c^2$. Thus, there are two possible mass-velocity relations, given by (4.3.3) and (4.3.4). which have the same approximation upto terms of order two of v as used in equation (4.4.3) below.

4.4 The Work Energy Theorem

Work done, $$W = \int_A^B \mathbf{F} \cdot d\boldsymbol{r} = \int_A^B \left(\frac{d\boldsymbol{p}}{dt}\right) \cdot d\boldsymbol{r} \qquad (4.4.1)$$

But, $\int (d\boldsymbol{p}/dt) \cdot d\boldsymbol{r} = \int \boldsymbol{v} \cdot d\boldsymbol{p} = \boldsymbol{v} \cdot \boldsymbol{p} - \int \boldsymbol{p} \cdot d\boldsymbol{v}$

$$= \boldsymbol{v} \cdot m\boldsymbol{v} - \int m\boldsymbol{v} \cdot d\boldsymbol{v}$$

$$= mv^2 - m_0 c^2 \exp(v^2/2c^2) + C' \qquad (4.4.2)$$

$$= mv^2 - mc^2 + C'$$

$$\approx m_0 v^2\left(1 + \frac{v^2}{2c^2}\right) - m_0 c^2\left(1 + \frac{v^2}{2c^2}\right) + C' \qquad (4.4.3)$$

$$= \frac{1}{2} m_0 v^2 - m_0 c^2 + \frac{1}{2} m_0 \frac{v^4}{c^2} + C'$$

$$\therefore W = \frac{1}{2} m_0 (v_B^2 - v_A^2) \qquad (4.4.4)$$

where $\frac{1}{2} m_0 v_A^2 = m_0 c^2 - \frac{1}{2} m_0 \frac{v^4}{c^2} + C'$ and $v_B = v$

that is, **change in Kinetic Energy = work done** (4.4.5)

Since the force field is central, we may take $\mathbf{F} = f(r)\mathbf{I}$ where $\mathbf{I}$ is a unit vector along the radial direction from centre of mass of the sub-system excluding the test mass m.

$$\therefore \mathbf{F} \cdot d\boldsymbol{r} = \mathbf{F} \cdot dr\mathbf{I} = f(r)dr$$

When A and B are points on the trajectory very near each other, the force may be approximated by a constant value say f_0

$$\therefore \int_{A}^{B} \mathbf{F} \cdot d\boldsymbol{r} = \int_{A}^{B} f_0 dr = f_0(r_B - r_A)$$

$$i.e.\, W = f_0(r - r_0) \qquad (4.4.6)$$

Equations (4.4.4) and (4.4.6) imply

$$\frac{1}{2} m_0(v^2 - v_0^2) = f_0(r - r_0)$$

this can be expressed as

$$v^2 = v_0^2 + (2f_0/m_0)(r - r_0) = f_1^2(r - r_1)$$

$$\text{or } v = f_1\sqrt{r - r_1} \qquad (4.4.7)$$

4.5 Principle of Least Time – Many Body Problem and the Inverse Square Law

In 1924, Louis de Broglie proposed that matter possesses waves as well as particle properties. The wave length λ of a particle of momentum $\boldsymbol{p}$ is given by $\lambda = h/|p|$ or $\boldsymbol{p} = \hbar \boldsymbol{k}$. In books on Quantum Mechanics, a simple wave is represented by $\psi = \psi_0 \text{ Exp.}(i/\hbar)(\mathbf{A} \cdot \boldsymbol{r} - \hbar\omega t)$ where $\mathbf{A}$ is normal to the wave front, $|\boldsymbol{k}| = 2\pi/\lambda$ is the wave number or propagation constant. A superposition of any such waves form a wave group or wave packet, moving with the velocity of the particle. Fermat's principle of least time is about the wave velocity/ phase velocity v_p of individual waves. It states that the time taken for a ray of light from a point A to a neighbouring point B is least

$$i.e.\, \delta \int_{A}^{B} \frac{ds}{v_p} = 0 \qquad (4.5.1)$$

We shall extend this principle to the wave groups, since this assumption is consistent with existing theory and serves as an alternative for the inverse square law, described in the next section. So, we state the principle of least time: A body of the n-body problem moves from A to B, taking the path of least time between A and B subject to the field constituted by the n bodies.

$$i.e.\ \delta \int_A^B \frac{ds}{v} = 0$$

where $v = f_1\sqrt{r - r_1}$ by the equation (4.4.7).

4.5.1 Proposition

The principle of least time/stationary time implies the inverse square law of gravitation. To prove the proposition, we find the trajectory of bodies in the Many Body Problem by solving $\delta I = 0$ where

$$I = \int (dr^2 + r^2 d\theta^2)^{½} v^{-1} = \int (1 + r^2\theta'^2)(r - r_1)^{-½} dr \qquad (4.5.2)$$

The Euler-Lagrange equation

$$(d/dr)(\partial f/\partial\theta') - \partial f/\partial\theta = 0 \qquad (4.5.3)$$

with $f = \sqrt{(1 + r^2\theta'^2)/(r - r_1)}$ gives

$(d/dr)(\partial f/\partial\theta') = 0$ since integrand as independent of θ

$$\therefore \partial f/\partial\theta' = \text{constant}$$

$$\sqrt{(r^2\theta')^2/[(1 + r^2\theta'^2)(r - r_1)]} = A$$

$$\therefore r^4\theta'^2 = A^2(1 + r^2\theta'^2)(r - r_1)$$

$$i.e.\quad \theta'^2[r^4 - A^2 r^2(r - r_1)] = A^2(r - r_1)$$

$$r^2\theta^2 = \frac{A^2(r - r_1)}{[r^2 - A^2(r - r_1)]}$$

$$\frac{1}{r^2\theta^{12}} = \frac{r^2 - A^2(r - r_1)}{[A^2(r - r_1)]} = \frac{r^2}{[A^2(r - r_1)]} - 1 \qquad (4.5.4)$$

Let $u = \frac{1}{r}$ or $\frac{du}{d\theta} = \frac{-1}{r^2} \cdot \frac{dr}{d\theta}$

$$i.e. \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{-1}{u} \frac{du}{d\theta}$$

$$\therefore \frac{1}{r^2\theta'^2} = \frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2 \qquad (4.5.5)$$

$$\therefore \frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2 = \frac{r^2}{A^2(r-r_1)} - 1$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{A^2(r-r_1)} \qquad (4.5.6)$$

Differentiating write θ

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{1}{A^2}\frac{-1}{(r-r_1)^2}\left(\frac{-1}{u^2}\right)\frac{du}{d\theta}$$

cancelling $2\frac{du}{d\theta}$ and letting $\frac{1}{2A^2} = \frac{MG}{\hbar_0^2}$

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{2A^2(r-r_1)^2u^2} = \frac{MG}{\hbar_0^2}\frac{r^2}{(r-r_1)^2} \qquad (4.5.7)$$

When $r_1 = 0$ this represents the conic

$$\frac{d^2u}{d\theta^2} + u = \frac{MG}{\hbar_0^2}$$

with a focus at the other end of the tube of force field of the particle.

If $f(r)$ is the law of force field for a central motion, then by a formula from classical mechanics, we have

$$\frac{d^2u}{d\theta^2} + u \equiv \frac{f(r)}{\hbar^2u^2} \qquad (4.5.8)$$

with the usual notations. Comparing (4.5.7) and (4.5.8) we have

$$\frac{f(r)}{\hbar^2u^2} = \frac{MG}{\hbar_0^2}\frac{r^2}{(r-r_1)^2}$$

$$\therefore f(r) = \frac{MG}{(r-r_1)^2} \qquad (4.5.9)$$

This represents the inverse square law; but r is the distance of the body from the centre of mass of the sub-system excluding the test body. Since $r - r_1 < r$, the distance $r - r_1$ must be the distance from the overall centre of mass of the whole system including the test mass m. Thus, Newton's inverse square law is always true and the law of force is

$$f(r) = \frac{MG}{[r]^2} \qquad (4.5.10)$$

Where $[r]$ is the radial distance of the test mass m from the common centre of mass. This equation (4.5.10) justifies us to use the centre of mass coordinates and define the corresponding potential ϕ_1 of Section 3 as

$$\phi_1 = \frac{MG}{[r]} \qquad (4.5.11)$$

Hence the principle of least/stationary time implies the inverse square law. When $r_1 \neq 0$ we proceed as follows: Expand the RHS of (4.5.7) by using binomial series

$$\therefore \frac{d^2u}{d\theta^2} + u = \left(\frac{MG}{\hbar_0^2}\right)(1 + 2r_1 u + 3r_1^2 u^2 + \cdots)$$

$$\frac{d^2u}{d\theta^2} + (1 - 2MGr_1 h_0^{-2})u = MGh_0^{-2}(1 + 3r_1^2 u^2 + \cdots) \qquad (4.5.12)$$

By letting $\varphi = (1 - 2MGr_1 h_0^{-2})\,\theta$, (4.5.12) can be re-written as

$$\frac{d^2u}{d\varphi^2} + u = MGh_0^{-2}\left(1 - 2MGr_1 h_0^{-2}\right)^{-1}(1 + 3r_1^2 u^2 + \cdots.) \qquad (4.5.13)$$

Discarding higher powers $u^3, u^4, \ldots$, we can rewrite (4.5.13) in the form of equation

$$\frac{d^2u}{d\theta^2} + u = \frac{MG}{h_0^2} + \frac{\mu_1 M}{4\pi}u^2 \qquad (4.5.14)$$

This shows that the modified Newton-Lorentz theory gives non-elliptic orbit as in GTR. It is also obtainable from the Schwarzschild metric

$$d\tau^2 = [1 - (2MG/R)]dt^2 - [(1 - (2MG/R))]^{-1} dR^2 - R^2 d\theta^2 - R^2 \sin^2\theta d\varphi^2 \qquad (4.5.15)$$

by redefining the parameters.

In the modified Newtonian dynamics, we are justified to use the retarded-potentials and retarded fields relative to centre of mass. On the other hand, the metric (4.5.15) has the special disadvantage that it has a singularity at $r = 2MG$ which can be removed according to Newton-Lorentz theory. Further, GTR does not consider Ampere's vector potential. Hence GTR cannot be considered as a generalization of modified Newton-Lorentz theory.

4.6 Non-relativistic Dirac's Equation, Faraday's Law and Lorentz Force Law

In Quantum Mechanics [14] the Hamiltonian operator for the motion of a charge q has the form

$$\mathrm{H} = \left(\boldsymbol{p} + \frac{qA}{c}\right) \cdot \boldsymbol{v} - q\phi + \boldsymbol{\beta} m_0 c^2 \qquad \text{(a)}$$

where the velocity operator $\boldsymbol{v} = \frac{dr}{d\tau} = \frac{1}{i\hbar}[\boldsymbol{r},\mathrm{H}] = \boldsymbol{\alpha} c$. By replacing $\boldsymbol{v}$ by $\boldsymbol{\alpha}$c, $\boldsymbol{p}$ $-\hbar i\nabla$ and H by $i\hbar\frac{\partial}{\partial t}$,

equation (a) becomes $i\hbar\frac{\partial}{\partial t} = \left[c\boldsymbol{\alpha} \cdot \left(-\hbar i\nabla + \frac{q}{c}\mathbf{A}\right) - q\phi + \boldsymbol{\beta} m_0 c^2\right]$

operating on ψ we get

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\boldsymbol{\alpha} \cdot \left(-\hbar i\nabla + \frac{q}{c}\mathbf{A}\right) - q\phi + \beta m_0 c^2\right]\psi \qquad \text{(b)}$$

This equation (b) is known as Dirac's equation where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are 4×4 matrices. Since we consider $m = m_0 \mathrm{Exp}\,\frac{v^2}{2c^2}$ instead of Lorentzian mass we can find a Hamiltonian by changing $\frac{q}{c}\mathbf{A}$ into $m\mathbf{A}_1$ and $q\phi$ into $m\phi_1$ in (a) and write a Hamiltonian according to Dirac's argument. The sign is chosen to get the Lorentz Force Law with the correct minus sign of $\mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial\mathbf{A}_1}{\partial t}$. Thus, we have the choice

for a mass particle $\mathrm{H}^* = (\boldsymbol{p} + m\mathbf{A}_1) \cdot \boldsymbol{v} - mc^2 - m\phi_1$ (4.6.1)(a)

for an electron $\mathrm{H}^* = (\boldsymbol{p} + q\mathbf{A}_2) \cdot \boldsymbol{v} - mc^2 - q\phi_2$ (4.6.1)(b)

so that for a mass particle

$$\mathrm{L}^* = \boldsymbol{p} \cdot \frac{\partial \mathrm{H}^*}{\partial \boldsymbol{p}} - \mathrm{H}^* = \boldsymbol{p} \cdot \boldsymbol{v} - \mathrm{H}^* = mc^2 + m\phi_1 - m\mathbf{A}_1 \cdot \boldsymbol{v} \qquad (4.6.2)$$

where $m = m_0 \mathrm{Exp}\,(v^2/2c^2)$

From the manner in which equation (b) is obtained from equation (a) it is clear that the non-relativistic Hamiltonian (4.6.1) (b) for an electron can be obtained by changing the signs of m_0 and q in equation (a); then the non-relativistic Dirac's equation obtainable from (4.6.1) (b), after dropping the subscript 1, is

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\boldsymbol{\alpha} \cdot \left(-\hbar i\nabla - \frac{q}{c}\mathbf{A}\right) + q\phi - \boldsymbol{\beta} m_0 c^2\right]\psi \qquad (4.6.3)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are four-matrices. A comparison between equations (b) and (4.6.3) reveals that both are non-relativistic equations.

The equation of motion corresponding to the Lagrangian L^* of equation (4.6.2), is

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \boldsymbol{v}}\right) = \nabla L^* \qquad (4.6.4)$$

$$i.e.\ \frac{d}{dt}(\boldsymbol{p} - m\mathbf{A}_1) = \nabla(mc^2) + m\nabla\phi_1 - m\nabla(\mathbf{A}_1 \cdot \boldsymbol{v}) \qquad (4.6.5)$$

where $m = m_0 \text{Exp}(v^2/2c^2)$

By taking $\nabla\ (mc^2) = 0$ the equation (4.6.5) becomes

$$\frac{d}{dt}(m^- \boldsymbol{v}) = -m\frac{d\mathbf{A}_1}{dt} - m\nabla\phi_1 + m\nabla(\mathbf{A}_1 \cdot \boldsymbol{v})$$

$$\approx -m\left[\left(\frac{\partial \mathbf{A}_1}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\mathbf{A}_1\right] - m\nabla\phi_1 + m[(\boldsymbol{v}\cdot\nabla)\mathbf{A}_1 + \boldsymbol{v}\times(\nabla \times \mathbf{A}_1)]$$

$$= -m\left(\nabla\phi_1 + \frac{\partial \mathbf{A}_1}{\partial t}\right) + m\boldsymbol{v}\times(\nabla \times \mathbf{A}_1)$$

$$i.e.\frac{d\boldsymbol{p}}{dt} = m^- \mathbf{E}_1 + m^- \boldsymbol{v} \times \mathrm{B}_1 \qquad (4.6.6)$$

This is Lorentz Force Law where

$$\mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial t} \qquad (4.6.7)$$

and

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 \qquad (4.6.8)$$

$$\text{Now } \nabla \times \mathbf{E}_1 = \frac{\partial}{\partial t}(\nabla \times \mathbf{A}_1) = \frac{-\partial \mathbf{B}_1}{\partial t} \qquad \text{i.e. } \nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t} \qquad (4.6.9)$$

This is Faraday's Law. Equation (4.6.1)(b) corresponding to an electron will not be treated here.

4.7 Poynting Theorem

The Poynting vector is defined by

$$\mathbf{S}_1 = \mathbf{E}_1 \times \mathbf{H}_1 \qquad (4.7.1)$$

and

$$u_1 = \frac{1}{2}\varepsilon_1 \mathbf{E}_1^2 + \frac{1}{2}\mu_1 \mathbf{H}_1^2 \qquad (4.7.2)$$

is defined as energy density.

Poynting's theorem [9] states that the work done on a mass-body by gravitational forces is equal to the decrease in energy stored in the field, less the energy that flowed out

$$i.e.\ \frac{dW}{dt} = \frac{\partial u_1}{\partial t} + \nabla \cdot \mathbf{S}_1 \qquad (4.7.3)$$

Proof : In the Lorentz Force Law given by equation (4.6.6)

$\mathbf{F} = m^{-} (\mathbf{E}_1 + \boldsymbol{v} \times \mathbf{B}_1)$ Suppose $m = \rho_1 dV$ and $\mathbf{F} = \boldsymbol{f}\, dV$ for a small mass m

$$\therefore \boldsymbol{f} = \rho_1^{-} (\mathbf{E}_1 + \boldsymbol{v} \times \mathbf{B}_1) \qquad (4.7.4)$$

By definition of work

$$dW = \boldsymbol{f} \cdot d\boldsymbol{r} \text{ or } \frac{dW}{dt} = \boldsymbol{f} \cdot \boldsymbol{v} \qquad (4.7.5)$$

$$i.e. - \frac{dW}{dt} = \rho_1 (\mathbf{E}_1 + \boldsymbol{v} \times \mathbf{B}_1) \cdot \boldsymbol{v} = \mathbf{E}_1 \cdot (\rho_1 \boldsymbol{v}) = \mathbf{E}_1 \cdot \mathbf{J}_1 \qquad (4.7.6)$$

Also $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_1) = \mathbf{H}_1 \cdot (\nabla \times \mathbf{E}_1) - \mathbf{E}_1 \cdot (\nabla \times \mathbf{H}_1)$

$$= -\mathbf{H}_1 \cdot \frac{\partial \mathbf{B}_1}{\partial t} - \mathbf{E}_1 \cdot \left(\mathrm{J}_1 + \frac{\partial \mathbf{D}_1}{\partial t} \right)$$

$$= -\mu_1 \mathbf{H}_1 \cdot \frac{\partial \mathbf{H}_1}{\partial t} - \mathbf{E}_1 \cdot \mathbf{J}_1 - \mathbf{E}_1 \cdot \frac{\partial \mathbf{E}_1}{\partial t}$$

$$i.e. \nabla \cdot \mathbf{S}_1 = \frac{-\partial}{\partial t} \left(\frac{1}{2} \varepsilon_1 \mathbf{E}_1 \cdot \mathbf{E}_1 + \frac{1}{2} \mu_1 \mathbf{H}_1 \cdot \mathbf{H}_1 \right) - \mathbf{E}_1 \cdot \mathbf{J}_1$$

$$= \frac{-\partial u_1}{\partial t} + \frac{dW}{dt} \text{ by using (4.7.2) and (4.7.6)}$$

4.8 Derivation of Momentum-Velocity Relation: $\boldsymbol{p}^2 = m^2 \boldsymbol{v}^2 + \hbar^2 \boldsymbol{k}^2$ and mass-energy relation: $\mathrm{E}^2 = m^2 c^4 + \hbar^2 \omega^2$ where $m = m_0 \exp(\boldsymbol{v}^2/2c^2)$

In the Compton-effect experiment, it was observed that, when a photon of energy $\hbar w = p_0 c$, strikes an electron of rest mass m_0, both will be deflected, and if the former makes angle θ and the latter an angle φ with the initial direction of photon, then by law of conservation of energy

$$(m - m_0)c^2 = \hbar (\omega - \omega') = (p_0 - p_1)c \qquad (4.8.1)$$

where p_0, p_1 are the initial and final momentum of the photon and m is the mass of the electron in motion.

$$\therefore\ p_0 - p_1 = (m - m_0)c \qquad (4.8.2)$$

Also, by the law of conservation of momentum, if $\boldsymbol{p}$ is the momentum of the electron, then

$$p_1 \cos\theta + p \cos\varphi = p_0$$

$$p_1 \sin\theta = p \sin\varphi$$

$$\therefore\ \boldsymbol{p}^2 = (p_0 - p_1)^2 + 2\, p_0\, p_1 (1 - \cos\theta) \qquad (4.8.3)$$

But $m^2 c^2 - m^2 \boldsymbol{v}^2 = (c^2 - v^2) m_0^2 \exp\left(\frac{\boldsymbol{v}^2}{c^2}\right)$

$$\approx m_0^2 c^2 \left(1 - \frac{\boldsymbol{v}^2}{c^2}\right)\left[1 + \frac{\boldsymbol{v}^2}{c^2} + \frac{\boldsymbol{v}^4}{2c^4}\right]$$

$$= m_0^2 c^2 \left(1 - \frac{\boldsymbol{v}^4}{2c^4} - \frac{\boldsymbol{v}^6}{3c^6}\right)$$

$$i.e.\ m^2 c^2 - m^2 v^2 \approx m_0^2 c^2 \left(1 - \frac{\boldsymbol{v}^4}{2c^4}\right) \qquad (4.8.4)$$

From (4.8.2) $mc = m_0 c + (p_0 - p_1)$

$$\therefore m^2 c^2 - \boldsymbol{p}^2 = [m_0 c + (p_0 - p_1)]^2 - (p_0 - p_1)^2 - 2\, p_0\, p_1 (1 - \cos\theta)$$

$$= m_0^2 c^2 + 2 m_0 (p_0 - p_1) c - 2\, p_0\, p_1\, (1 - \cos\theta)$$

$$= m_0^2 c^2 + 2 m_0 (m - m_0) c^2 - 2\, p_0\, p_1 (1 - \cos\theta) \text{ by } (4.8.2)$$

But $m - m_0 = m\,[1 - \exp(-\boldsymbol{v}^2/2c^2)] \approx (m\,\boldsymbol{v}^2/2c^2)$

$$\therefore\ m^2 c^2 - \boldsymbol{p}^2 = m_0^2 c^2 + 2 m_0 c^2 (m v^2/2c^2) - 2\, p_0\, p_1 (1 - \cos\theta)$$

$$i.e. m^2 c^2 - \boldsymbol{p}^2 = m_0^2 c^2 + m_0\, m\, \boldsymbol{v}^2 - 2\, p_0\, p_1\, (1 - \cos\theta) \qquad (4.8.5)$$

As a first approximation we may take $\boldsymbol{p} \approx m\,\boldsymbol{v}$ ∴ LHS of equations (4.8.4) and (4.8.5) are equal.

∴ RHS must be approximately equal.

$$\therefore m_0^2 c^2 \left(1 - \frac{\boldsymbol{v}^4}{2c^4}\right) \approx m_0^2 c^2 + m_0 m\boldsymbol{v}^2 - 2 p_0 p_1 (1 - \cos \theta)$$

$$\therefore 2 p_0 p_1 (1 - \cos \theta) \approx m_0 m\boldsymbol{v}^2 + \frac{1}{2} m_0^2 v^4 c^{-2}$$

$$\approx m_0 \boldsymbol{v}^2 \left[m + \left(\frac{m_0 \boldsymbol{v}^2}{2c^2}\right)\right]$$

$$= m_0^2 \boldsymbol{v}^2 \left(\exp\cdot\frac{\boldsymbol{v}^2}{2c^2} + \frac{\boldsymbol{v}^2}{2c^2}\right)$$

$$\approx m_0^1 \boldsymbol{v}^2 \left(1 + \frac{\boldsymbol{v}^2}{2c^2} + \frac{\boldsymbol{v}^2}{2c^2}\right)$$

$$= m_0^2 \boldsymbol{v}^2 \left(1 + \frac{\boldsymbol{v}^2}{c^2}\right)$$

$$\approx m_0^2 \boldsymbol{v}^2 \exp\left(\frac{\boldsymbol{v}^2}{c^2}\right)$$

$$= m^2 \boldsymbol{v}^2 \qquad (4.8.6)$$

using (4.8.6) in (4.8.3) the latter equation becomes

$$\boldsymbol{p}^2 = (p_0 - p_1)^2 + m^2 \boldsymbol{v}^2$$

which is of the form

$$\boldsymbol{p}^2 = m^2 \boldsymbol{v}^2 + \hbar^2 \boldsymbol{k}^2 \qquad (4.8.7)$$

i.e. momentum consists of a mechanical part $m\boldsymbol{v}$ and a quantum-mechanical part/electromagnetic part $i\hbar\boldsymbol{k}$; so, we can write

$$\boldsymbol{p}^* = m\boldsymbol{v} \pm i\hbar\boldsymbol{k} = m\boldsymbol{v} \pm iq\mathrm{A}$$

$$= m\left(\boldsymbol{v} \pm \frac{iq}{m}\mathbf{A}\right) \qquad (4.8.8)(a)$$

$$= m(\boldsymbol{v} \pm i\boldsymbol{v}_p) \qquad (4.8.8)(b)$$

Equation (4.8.4) can be rewritten as

$$\left[m^2 c^2 + \hbar^2 \boldsymbol{k}^2 + \frac{1}{2} m_0^2 \frac{v^4}{c^2}\right] - (m^2 \boldsymbol{v}^2 + \hbar^2 \boldsymbol{k}^2) = m_0^2 c^2$$

which is of the form $E^2c^2 - \boldsymbol{p}^2 = m_0^2\, c^2$ or $E^2 - \boldsymbol{p}^2c^2 = m_0^2\, c^4$

where $E^2 = m^2c^4 + \hbar^2\omega^2$ and $\hbar^2\omega^2 = \hbar^2\boldsymbol{k}^2c^2 + \frac{1}{2}m_0^2v^4$

∴ Energy consists of an inertial part mc^2 and a quantum-mechanical part/electromagnetic part $i\hbar\omega$.

It is thus possible to take $\bar{E} = mc^2 - i\hbar\omega = mc^2 - i\hbar(\partial/\partial t)\log\psi)$

$$i.e.\,\bar{E}\psi = mc^2\psi - i\hbar\left(\frac{\partial}{\partial t}\right)\psi \qquad (4.8.9)$$

$$\text{and } \bar{\boldsymbol{p}}\psi = m\boldsymbol{v} + i\hbar\boldsymbol{k} = m\boldsymbol{v} + i\hbar\left(\frac{\partial}{\partial \boldsymbol{r}}\right)(\log\psi)$$

$$i.e.\,\bar{\boldsymbol{p}}\psi = m\boldsymbol{v}\psi + i\left(\frac{\partial}{\partial \boldsymbol{r}}\right)\psi \qquad (4.8.10)$$

5. Applications to Fluid dynamics

In free space, we took ε_1 and μ_1 as the parameters of free space. In fluid motion, even within an elemental volume dV, there will be many particles thickly packed together; hence these parameters will have to be modified to some average value ε'_1 and μ'_1. This is to make allowance for the interaction of fluid molecules, such as friction, gravity, viscosity. Hence ε'_1 and μ'_1 take the roles of ε_1 and μ_1 and $\varepsilon'_1\,\mu'_1\,c^2$ may be less than unity.

5.1 Biot - Savart Law

In the Lorentz Force Law

$$-\mathbf{F}_1 = m\,(\mathbf{E}_1 + \boldsymbol{v} \times \mathbf{B}_1) \qquad (5.1.1)$$

the second term $m\boldsymbol{v} \times \mathbf{B}_1$ gives a measure of the force/torque due to the gravitational vortex/flux density $\mathbf{B}_1$ and is also a measure of vortex motion, in the case of fluids in motion. Consider an elemental volume

$$dV = A|d\boldsymbol{\ell}| = A\,|d\mathbf{r}|$$

of uniform cross-section A and length $|d\boldsymbol{\ell}|$. The rate of flow of fluid volume is $\frac{dV}{dt} = A\left|\frac{dl}{dt}\right| = A\,|\boldsymbol{v}|$ and the rate of mass crossing A per unit time is

$$\rho_1\frac{dV}{dt} = \rho_1 A\,|\boldsymbol{v}| = A|\rho_1\boldsymbol{v}| = A\,|\mathbf{J}_1| = I$$

$$\therefore I d\boldsymbol{\ell} = I d\boldsymbol{r} = \rho_1\frac{dV}{dt}\,d\boldsymbol{r} = m\boldsymbol{v} = \rho_1\frac{d\boldsymbol{r}}{dt}\,dV = \mathbf{J}_1 dV \qquad (5.1.2)$$

where $\rho_1 dV = m$, is the mass contained in dV and

$$J_1 = \rho_1 \boldsymbol{v} \qquad (5.1.3)$$

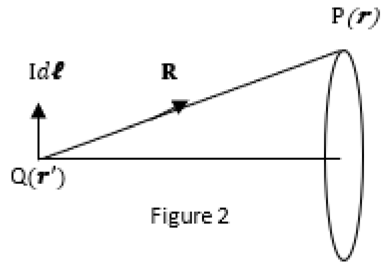

Figure 2

Denoting vortex/flux density contributed by dV situated at $Q(\boldsymbol{r}')$, at the field point $P(\boldsymbol{r})$ by $d\mathbf{B}_1$ we have

$$d\mathbf{B}_1 = \frac{\boldsymbol{v} \times \mathbf{E}_1}{c^2} = \frac{\boldsymbol{v} \times m\widehat{\mathbf{R}}}{c^2 4\pi\varepsilon_1 \mathrm{R}^2}$$

$$= \frac{\mu_1 \mathrm{I}d\boldsymbol{\ell} \times \widehat{\mathbf{R}}}{4\pi \mathrm{R}^2} \qquad (5.1.4)$$

$$\therefore \ \mathbf{B}_1 = \int_{\mathrm{C}} \frac{\mu_1 \mathrm{I}d\boldsymbol{\ell} \times \widehat{\mathbf{R}}}{4\pi \mathrm{R}^2} \qquad (5.1.5)$$

which can be re-written as

$$\mathbf{B}_1 = \int_{\mathrm{V}'} \frac{\mu_1 \mathbf{J}_1(\mathbf{r}') \times \widehat{\mathbf{R}}\, d\mathrm{V}}{4\pi \mathrm{R}^2} \qquad (5.1.6)$$

where C is arc, of which $\mathrm{I}d\boldsymbol{\ell}$ is a part and V' is the volume of mass forming C

Equations (5.1.5) and (5.1.6) are known as Biot-Savart Law, the former as a line-integral and the latter as a volume integral.

5.2 Ampere's Vector Potential $\mathbf{A}_1$

By defining

$$\mathbf{A}_1(\boldsymbol{r}) = \frac{\mu_1}{4\pi} \int_{\mathrm{C}} \frac{\mathrm{I}d\ell}{|\boldsymbol{r} - r'|} \qquad (5.2.1)$$

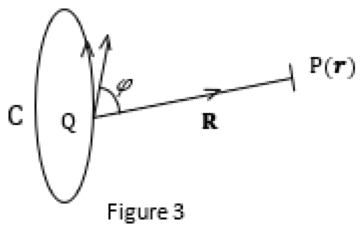

Figure 3

$$\text{or} \quad \mathbf{A}_1(\boldsymbol{r}) = \frac{\mu_1}{4\pi}\int_{V'} \frac{\mathbf{J}_1(\boldsymbol{r}')\,dV'}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad (5.2.2)$$

where V' is the volume of the flow/current loop C under consideration, we can prove that

$$\mathbf{B}_1 = \text{Curl } \mathrm{A}_1 \qquad (5.2.3)$$

Proof:

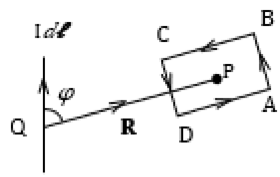

Figure 4

Consider an elemental fluid flow $d\boldsymbol{\ell}$ at Q ($\boldsymbol{r}'$); we seek its contribution to $\mathbf{A}_1$ at P. Take a square loop A-B-C-D-A centred at P of sides $2h$ with two sides AB and CD perpendicular to QP and the remaining two sides parallel to QP.

$$\therefore\ d\boldsymbol{\ell} = d\ell\cos\varphi\,\boldsymbol{i} + d\ell\sin\varphi\,j$$

where (i, j) is a pair of basis vectors i being along QP and j orthogonal to it in the plane of $d\ell$ and $\mathbf{R}$ and φ is the inclination of loop element with QP.

Area of square, $\Delta S = 4h^2$

$$\boldsymbol{i} = \hat{\mathbf{R}}, \overrightarrow{QP} = \mathbf{R} = \mathrm{R}\boldsymbol{i} = \boldsymbol{r} - \boldsymbol{r}'$$

$$\therefore\ \Delta\mathbf{A}_1 = \frac{\mu_1 I d\boldsymbol{\ell}}{4\pi R}$$

$$\overrightarrow{AB} = 2\,h\boldsymbol{j}, \overrightarrow{CD} = -2h\boldsymbol{j}$$

$$\overrightarrow{BC} = -2h\boldsymbol{i}, \overrightarrow{DA} = 2h\boldsymbol{i}$$

$$\oint_{ABCDA} \Delta A_1 \cdot d\boldsymbol{r} = \frac{\mu_1 I d\boldsymbol{\ell}}{4\pi}\left[\frac{2h \sin\varphi}{R + h} - \frac{2h\cos\varphi}{\sqrt{R^2+h^2}} - \frac{2h \sin\varphi}{R-h} + \frac{2h\cos\varphi}{\sqrt{R^2+h^2}}\right] = \mu_1 \frac{I d\boldsymbol{\ell} \sin\varphi(-4h^2)}{4\pi(R^2-h^2)}$$

$$\lim_{h\to 0}\left[\frac{\oint \Delta\mathbf{A}_1 \cdot d\boldsymbol{r}}{\Delta S}\right] = \lim_{h\to 0}\left[\frac{-\mu_1 I d\boldsymbol{\ell}\sin\varphi}{4\pi(R^2-h^2)}\right] = \frac{-\mu_1 I d\boldsymbol{\ell}\sin\varphi}{4\pi R^2}$$

$$i.e.\ \Delta\left[\oint \frac{\mathbf{A}_1 \cdot d\boldsymbol{r}}{\Delta S}\right]\mathbf{k} = \left[\frac{\mu_1\, I d\boldsymbol{\ell} \times \widehat{R}}{4\pi R^2}\right]\boldsymbol{k} \text{ where } \boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$$

$$i.e.\ \Delta(\text{curl}\ \mathbf{A}_1) = \Delta\mathbf{B}_1$$

$\therefore\ \mathbf{B}_1 = \nabla \times \mathbf{A}_1$ which is equation (5.2.3) and

$$\text{Div}\ \mathbf{B}_1 = \text{Div}\ (\nabla \times \mathbf{A}_1) \equiv 0 \qquad (5.2.4)$$

Since $\mathbf{A}_1$ at each point on the arc C is in the direction of $I d\boldsymbol{\ell}$ and the *Pfaffian* differential form $\mathbf{A}_1 \cdot d\boldsymbol{\ell}$ or $\mathbf{A}_1 \cdot \mathbf{dr}$ is integrable *iff* $\mathbf{A}_1 \cdot \text{curl}\ \mathbf{A}_1 \equiv 0$ we must have $\mathbf{A}_1 \cdot (\nabla \times \mathbf{A}_1) = 0$ i.e. lines of $\mathbf{A}_1$ are orthogonal to lines of $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ and hence parallel to C.

5.3 Induced vortex field on the axis of a flow-loop

Take any point P on the flow-loop in the form of a circle and any point Q on its axis. The induced vortex-field $d\mathbf{B}_Q$ due to an elemental flow $d\boldsymbol{\ell}$ at P (on the loop of fluid flow), is given by

$$d\mathbf{B}_Q = \frac{\mu_1 I\, d\boldsymbol{\ell} \times (-\mathbf{R})}{4\pi R^3} = \frac{\mu_1 I\, \mathbf{R} \times d\boldsymbol{\ell}}{4\pi R^3} = \frac{\mu_1 I d\boldsymbol{\ell}\boldsymbol{n}}{4\pi R^2}$$

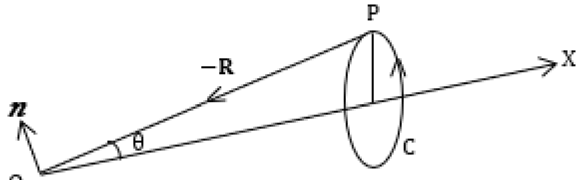

Figure 5

where $\boldsymbol{n}$ is a unit normal vector at Q perpendicular to QP. Considering the contributions of all such elements $d\boldsymbol{\ell}$ on the circular loop we see that the component normal to QX vanishes. Therefore $\mathbf{B}_Q$ has only an axial component along QX/XQ.

$$\therefore\ \mathbf{B}_Q = \frac{\mu_1 I \cos\left(\frac{\pi}{2} + \theta\right)}{4\pi R^2}(R\sin\theta)\boldsymbol{i}\int_0^{2\pi} d\varphi \quad \text{along QX}$$

since radius of loop is $\mathrm{R}\sin\theta$, $\boldsymbol{n}$ makes angle $\frac{\pi}{2}+\theta$ with QX and the central angle of circular loop is chosen as φ, measured from a fixed diameter.

$$\therefore \mathbf{B}_{\mathrm{Q}} = \frac{-\mu_1 \mathrm{IR}\sin^2\theta \cdot 2\pi}{4\pi \mathrm{R}^2}\,\boldsymbol{i}$$

$$= \frac{-\mu_1 \mathrm{I}}{2\,\mathrm{R}}\sin^2\theta\,\boldsymbol{i}$$

$$= \frac{-\mu_1\,\mathrm{I}\sin^3\theta}{2\,(\mathrm{R}\sin\theta)}\,\boldsymbol{i}$$

$$\mathbf{B}_{\mathrm{Q}} = \frac{-\mu_1\,\mathrm{I}\,\sin^3\theta}{2r_x}\,\boldsymbol{i} \qquad (5.3.1)$$

where r_x is the radius of the loop.

∴At the centre of the loop.

$$\mathbf{B}_{\mathrm{Q}} = \frac{-\mu_1\,\mathrm{I}}{2r_x}\boldsymbol{i} \qquad (5.3.2)$$

The negative sign shows that the circular vortex flow along C, induced by a flow element at Q, opposes the flow element at Q by a counter-induced vortex field $\mathbf{B}_{\mathrm{Q}} = -\frac{\mu_1 \mathrm{I}\sin^3\theta}{2r_x}$, causing to stop the flow at Q. This situation is analogous to the Faradays' law and Lenz's law in elector-magnetism: A changing magnetic field induces a current in a closed loop of wire placed in the field (Faraday's Law) and the direction of the induced emf is such that any current that it produces, tends to oppose the change of flux (Lenz's law).

6. Gravitational Wave Equations

From Section 4.6 we have

$$\mathbf{E}_3 = -\nabla\Phi_3 - \frac{\partial \mathbf{A}_3}{\partial t}$$

$$\mathbf{B}_3 = \nabla \times \mathbf{A}_3$$

where $\phi_3 = \phi_1 \pm \frac{iq}{m}\phi_2$ and $\mathbf{A}_3 = \mathbf{A}_1 \pm \frac{iq}{m}\,\mathbf{A}_2$.To restrict our attention to the gravitational field, we take $\Phi_2 = 0, \mathbf{A}_2 = 0$ so that

$$\mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial \mathrm{t}} \qquad (6.1.1)$$

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 \qquad (6.1.2)$$

$$\therefore \text{ Div } \mathbf{E}_1 = -\nabla^2\phi_1 - \frac{\partial}{\partial t}(\text{Div } \mathbf{A}_1)$$

$$i.e.\ \frac{\rho_1}{\varepsilon_1} = -\nabla^2\phi_1 - \frac{\partial^2}{\partial t^2}\left(\frac{-\phi_1}{c^2}\right)$$

by assuming equation of continuity (gauge condition) :

$$\frac{\partial\phi_1}{\partial t} + c^2 \text{Div } \mathbf{A}_1 = 0 \qquad (6.1.3)$$

and Gauss' Law.

$$\text{i. e. } \frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t^2} - \nabla^2\phi_1 = \frac{\rho_1}{\varepsilon_1} \qquad (6.1.4)$$

Changing $\mathbf{E}_1$ to $\mathbf{E}_1^-$ and ρ_1 to ρ_1^- we get

$$\frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t^2} - \nabla^2\phi_1 = \frac{\rho_1^-}{\varepsilon_1} \qquad (6.1.4')$$

where ϕ_1 in equation (6.1.4$'$) is to be replaced by $(-\phi_1)$ for the attractive fields so that we have

$$\frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t^2} - \nabla^2\phi_1 = \left|\frac{\rho_1^-}{\varepsilon_1}\right|$$

From Ampere's law

$$\nabla \times \mathbf{H}_1 = \rho_1\boldsymbol{v} + \frac{\partial \mathbf{D}_1}{\partial t}$$

and $\mathbf{B}_1 = \mu_1\mathbf{H}_1 = \nabla \times \mathbf{A}_1$

$$\nabla \times \mathbf{B}_1 = \nabla \times (\nabla \times \mathbf{A}_1)$$

$$= \nabla(\nabla\cdot\mathbf{A}_1) - \nabla^2\mathbf{A}_1$$

$$\therefore \mu_1\left(\rho_1\mathbf{v} + \frac{\partial \mathbf{D}_1}{\partial t}\right) = \nabla(\nabla\cdot\mathbf{A}_1) - \nabla^2\mathbf{A}_1 \qquad (6.1.5)$$

$$\therefore \mu_1\rho_1\boldsymbol{v} + \mu_1\frac{\partial \mathbf{D}_1}{\partial t} = \nabla\left(-\frac{1}{c^2}\frac{\partial\phi_1}{\partial t}\right) - \nabla^2\mathbf{A}_1 \qquad (6.1.6)$$

From (6.1.1) $\frac{\partial \mathbf{E}_1}{\partial t} = -\nabla\frac{\partial\phi_1}{\partial t} - \frac{\partial^2\mathbf{A}_1}{\partial t^2}$. Multiplying by $\frac{1}{c^2}$ and rearranging

$$-\nabla\left(\frac{1}{c^2}\ \frac{\partial\varphi_1}{\partial t}\right) = \frac{1}{c^2}\ \frac{\partial^2\mathbf{A}_1}{\partial t^2} + \frac{1}{c^2}\ \frac{\partial\mathbf{E}_1}{\partial t}$$

$$= \frac{1}{c^2}\ \frac{\partial^2\mathbf{A}_1}{\partial t^2} + \mu_1\varepsilon_1\ \frac{\partial\mathbf{E}_1}{\partial t} \qquad (6.1.7)$$

By using (6.1.7) in (6.1.6) we have

$$\mu_1\ \rho_1\boldsymbol{v} + \mu_1\varepsilon_1\frac{\partial\mathbf{E}_1}{\partial t} = \left(\frac{1}{c^2}\ \frac{\partial^2\mathbf{A}_1}{\partial t^2} + \mu_1\varepsilon_1\frac{\partial\mathbf{E}_1}{\partial t}\right) - \nabla^2\mathbf{A}_1$$

$$\text{i.e.}\quad \frac{1}{c^2}\ \frac{\partial^2\mathbf{A}_1}{\partial t^2} - \nabla^2\mathbf{A}_1 = \mu_1\rho_1\boldsymbol{v} \qquad (6.1.8)$$

Changing $\mathbf{E}_1$ to $\mathbf{E}_1^-$ and ρ_1 to ρ_1^- we get

$$\frac{1}{c^2}\ \frac{\partial^2\mathbf{A}_1}{\partial t^2} - \nabla^2\mathbf{A}_1 = \mu_1\rho_1^-\boldsymbol{v} = -\mu_1\rho_1\boldsymbol{v} \qquad (6.1.8')$$

where $\mathbf{A}_1$ is to be replaced by $(-\mathbf{A}_1)$ for the attractive fields.

Equation (6.1.4) and (6.1.8) show that both Φ_1 and $\mathbf{A}_1$ satisfy the inhomogeneous wave-equations, as in electro-magnetism.

When Φ_1 and $\mathbf{A}_1$ are independent of time, (6.1.4') and (6.1.8')become the Poisson equations [8,9,13] for repulsive/attractive gravitational fields:-

$$\nabla^2\phi_1 + \left|\frac{\rho_1^-}{\varepsilon_1}\right| = 0 \quad \text{and} \quad \nabla^2\mathbf{A}_1 + \mu_1\mathbf{J}_1 = \mathbf{0} \quad \text{where } \mathbf{J}_1 = |\rho_1^-|\boldsymbol{v}.$$

These have solutions $\pm\ \phi_1, \pm\mathbf{A}_1$ where

$$\phi_1(\boldsymbol{r}) = \frac{\mu_1}{4\pi\varepsilon_1}\iiint_{V'}\frac{\rho_1(\boldsymbol{r}')dV'}{R} \qquad (6.1.9)$$

$$\mathbf{A}_1(\boldsymbol{r}) = \frac{\mu_1}{4\pi}\iiint_{V'}\frac{J_1(\boldsymbol{r}')dV'}{R} \qquad (6.1.10)$$

From these we write the solutions of (6.1.4) and (6.1.8) in the form:

$$\phi_1(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_1}\iiint_{V'}\frac{\rho_1(\boldsymbol{r}',t_r)dV'}{R(t_r)} \qquad (6.1.11)$$

$$\mathbf{A}_1(\boldsymbol{r},t) = \frac{\mu_1}{4\pi}\iiint_{V'}\frac{J_1(\boldsymbol{r}',t_r)}{R(t_r)}dV' \qquad (6.1.12)$$

where the integrals are evaluated at the retarded time t_r; it is clear that distinct elemental volume dV' of V' will have different retarded time during the evaluation. It can be shown that these retarded potentials satisfy (6.1.4) and (6.1.8) and are the generalisations of all the previous expressions for $(\phi_1, \mathbf{A}_1)$ given by equations (4.1.9), (4.2.8) and (4.2.9). By substituting equations (6.1.11) and (6.1.12) in we get

$$\mathbf{E}_1(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_1}\iiint_{V'}\left[\frac{\rho_1(\boldsymbol{r}',t_r)\widehat{\mathrm{R}}}{\mathrm{R}^2} + \frac{\dot{\rho}_1(\boldsymbol{r}',t_r)\widehat{\mathbf{R}}}{\mathrm{R}c} - \frac{\dot{\mathbf{J}}(\boldsymbol{r}',t_r)}{\mathrm{R}c^2}\right]d\mathrm{V}' \qquad (6.1.13)$$

$$\mathbf{B}_1(\boldsymbol{r},t) = \frac{\mu_1}{4\pi}\iiint_{V'}\left[\frac{\mathbf{J}_1(\boldsymbol{r}',t_r)}{\mathrm{R}^2} + \frac{\dot{\mathbf{J}}_1(\boldsymbol{r}',t_r)}{\mathrm{R}c}\right] \times \widehat{\mathrm{R}}d\mathrm{V}' \qquad (6.1.14)$$

where $\mathbf{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $\widehat{\mathrm{R}} = \mathbf{R}/\mathrm{R}$. Similarly, we can find the solution in the electro-magnetic case by replacing the subscript 1 by 2. Equations (6.1.11) to (6.1.14) give the most general expressions for $\Phi_1, \mathbf{A}_1, \mathbf{E}_1$, and $\mathbf{B}_1$.

7 Derivation of Orbit of Planetary Motion

We shall derive the equation of planetary motion by two methods as follows.

7.1 By using Newton's Law of Motion

We have

$$\frac{d\boldsymbol{p}}{dt} = \mathbf{F}_1 \qquad (7.1.1)$$

As in approximation

$$\frac{d}{dt}(m\boldsymbol{v}) = m\boldsymbol{E}_1^- \text{ where } \boldsymbol{E}_1^- = \frac{-\mathrm{M}\mathbf{R}}{4\pi\varepsilon_1\mathrm{R}^3} = \frac{-\mathrm{M}\mathbf{I}}{4\pi\varepsilon_1\mathrm{R}^2}$$

$$i.e.\ m\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v}\frac{dm}{dt} = m\boldsymbol{E}_1^- \qquad (7.1.2)$$

$$\therefore \boldsymbol{v} \times (d\boldsymbol{v}/dt) = \boldsymbol{v} \times \boldsymbol{E}_1^- \qquad (7.1.3)$$

But from $m = m_0\mathrm{Exp}(v^2/2c^2)$ we have

$$\frac{dm}{dt} = m(\boldsymbol{v}/c^2)\cdot(d\boldsymbol{v}/dt) \ \therefore\ (7.1.2) \Rightarrow$$

$$\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v}[(\boldsymbol{v}/c^2)\cdot(d\boldsymbol{v}/dt)] = \boldsymbol{E}_1^-$$

$$\text{i.e.}\quad \frac{d\boldsymbol{v}}{dt} + (\boldsymbol{v}/c^2)\times(\boldsymbol{v}\times d\boldsymbol{v}/dt) + (\boldsymbol{v}^2/c^2)(d\boldsymbol{v}/dt) = \boldsymbol{E}_1^-$$

$$i.e.\,(d\boldsymbol{v}/dt) + [\boldsymbol{v}\times(\boldsymbol{v}\times\mathrm{E}_1)]/c^2 \approx \boldsymbol{E}_1^- = \frac{-\mathrm{M}\mathbf{I}}{4\pi\varepsilon_1\mathrm{R}^2} \qquad (7.1.4)$$

$$\text{discarding } \frac{v^2}{c^2} \approx 0$$

$$v \times E_1^- = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dot{\mathrm{R}} & \mathrm{R}\dot{\theta} & 0 \\ \frac{-\mathrm{M}}{4\pi\varepsilon_1 \mathrm{R}^2} & 0 & 0 \end{vmatrix} = \frac{\mathrm{M}\mathrm{R}\dot{\theta}\mathbf{K}}{4\pi\varepsilon_1 \mathrm{R}^2}$$

$$v \times (v \times E_1^-) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dot{\mathrm{R}} & \mathrm{R}\dot{\theta} & 0 \\ 0 & 0 & \frac{\mathrm{M}\dot{\theta}}{4\pi\varepsilon_1 \mathrm{R}} \end{vmatrix} = \frac{\mathrm{M}\dot{\theta}(\mathrm{R}\dot{\theta}\mathbf{I} - \dot{\mathrm{R}}\mathbf{J})}{4\pi\varepsilon_1 \mathrm{R}} = \frac{\mathrm{M}\dot{\theta}^2\mathbf{I}}{4\pi\varepsilon_1} - \frac{\mathrm{M}\dot{\mathrm{R}}\dot{\theta}}{4\pi\varepsilon_1 \mathrm{R}}\mathbf{J} \quad (7.1.5)$$

Since $\frac{dv}{dt} = (\ddot{\mathrm{R}} - \mathrm{R}\dot{\theta}^2)\mathbf{I} + \frac{1}{\mathrm{R}}[d(\mathrm{R}^2\dot{\theta})/dt]\mathbf{J}$,(7.1.4) and (7.1.5) imply

$$\ddot{\mathrm{R}} - \mathrm{R}\dot{\theta}^2 = \frac{-\mathrm{M}}{4\pi\varepsilon_1 \mathrm{R}^2} - \frac{\mathrm{M}\dot{\theta}^2}{4\pi\varepsilon_1 c^2} \qquad (7.1.6)$$

$$\text{and } \frac{d(\mathrm{R}^2\dot{\theta})}{\mathrm{R}dt} = \frac{\mathrm{M}\dot{\mathrm{R}}\dot{\theta}}{4\pi\varepsilon_1 \mathrm{R}c^2} \qquad (7.1.7)$$

The latter can be re-written as

$$\frac{d(\mathrm{R}^2\dot{\theta})}{\mathrm{R}^2\dot{\theta}} = \frac{\mathrm{M}d\mathrm{R}}{4\pi\varepsilon_1 c^2\mathrm{R}^2}$$

Integrating we get

$$\mathrm{R}^2\frac{\mathrm{d}\theta}{\mathrm{dt}} = h_0 \text{ Exp } (-\mathrm{M}/4\pi\varepsilon_1 c^2\mathrm{R}) = h_0\text{Exp}(-\phi_1/c^2) \text{ or } h_0\text{Exp}\left(\frac{-\mathrm{MG}}{c^2\mathrm{R}}\right) \qquad (7.1.8)$$

As an approximation

$$\mathrm{M}\mathrm{R}^2\dot{\theta} \approx \mathrm{H}_0 \text{ or } \mathrm{R}^2\dot{\theta} \approx h_0 \qquad (7.1.9)$$

Letting $u = \frac{1}{\mathrm{R}}$ We have $\dot{\mathrm{R}} = \left(\frac{-1}{u^2}\right)(du/d\theta)\dot{\theta} = -h_0(du/d\theta)$

$$\ddot{\mathrm{R}} = -h_0(d^2u/d\theta^2)\dot{\theta}$$

$$\ddot{\mathrm{R}} - \mathrm{R}\dot{\theta}^2 = -h_0(d^2u/d\theta^2)h_0u^2 - (1/u)(h_0u^2)^2 = -h_0^2u^2[(d^2u/d\theta^2) + u]$$

Now (7.1.6) can be re-written as

$$-h_0^2 u^2 \left(\frac{d^2u}{d\theta^2} + u\right) = \frac{-\mathrm{M}u^2}{4\pi\varepsilon_1} - \frac{\mathrm{M}\cdot(h_0 u^2)^2}{4\pi\varepsilon_1 c^2}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{\mathrm{M}}{h_0^2 \cdot 4\pi\varepsilon_1} + \frac{\mathrm{M}u^2}{4\pi\varepsilon_1 c^2}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{\mathrm{MG}}{h_0^2} + \frac{\mathrm{M}u^2}{4\pi\varepsilon_1 c^2} = \frac{\mathrm{MG}}{h_0^2} + \frac{\mu_1 \mathrm{M}}{4\pi} u^2 \qquad (7.1.10)$$

This is the equation for planetary motion [20] already found in equation (4.5.14).

By replacing ε_1 by ε_2, μ_1 by μ_2, M by Q$|q/m|$ and Mm by |Qq| where (q/m) is nearly a constant for small velocities and replacing $\mathrm{G} = (4\pi\varepsilon_1)^{-1}$ by $(4\pi\varepsilon_2)^{-1}$ we get the corresponding equation (7.1.10) for the motion of an electron in Electro-magnetic field. Thus the modified Newton-Lorentz mechanics can handle motion of both mass particle and electron. On the other hand, GTR being a theory of gravitation cannot handle motion of electrons and is applicable to mass particle only.

7.2 By using Modified Newton-Lorentz Equation

$$\text{We have} - \frac{d\boldsymbol{p}}{dt} = m\mathbf{E}_1 + m\boldsymbol{v} \times \mathbf{B}_1 \qquad (7.2.1)$$

$$\text{where}, \mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial t} \qquad (7.2.2)$$

$$\mathbf{B}_1 = \boldsymbol{\nabla} \times \mathbf{A}_1 \qquad (7.2.3)$$

where, $\phi_3 = \phi_1 \pm \frac{iq}{m}\phi_2$ and $\mathbf{A}_3 = \mathbf{A}_1 \pm \frac{iq}{m}\mathbf{A}_2$

we shall exclude $\frac{q}{m}\mathbf{A}_2$ and ϕ_2 for the purely gravitational fields.

$$\therefore \mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial t} \approx -\nabla\phi_1 = \mathrm{E}_1\,\mathbf{I} \text{ and } \mathbf{B}_1 = \frac{(-\boldsymbol{v}) \times \mathbf{E}_1^-}{c^2} \text{ by } (4.2.10)$$

Here the minus is prefixed to $\boldsymbol{v}$, since the test particle of mass m is moving, whereas in (4.1.9), $\frac{\boldsymbol{v}\times\mathbf{E}_1}{c^2}$ is the field due to source mass M moving with the velocity $\boldsymbol{v}$.

$\therefore$ (4.6.6) becomes

$$-\frac{d\boldsymbol{p}}{dt} = m\,\mathbf{E}_1 + m\boldsymbol{v} \times (-\boldsymbol{v} \times \mathbf{E}_1^-)c^{-2}$$

$$i.e. - m\frac{d\boldsymbol{v}}{dt} \approx m\mathbf{E}_1 + m\boldsymbol{v} \times (-\boldsymbol{v} \times \mathbf{E}_1^-)c^{-2}$$

taking $\dot{m} \approx 0$ we get (7.1.4), viz

$\therefore -\frac{d\boldsymbol{v}}{dt} \approx \mathbf{E}_1 + \boldsymbol{v} \times (\boldsymbol{v} \times \mathbf{E}_1)c^{-2}$ which is equation (7.1.4) since $\mathbf{E}_1^- = -\mathbf{E}_1$.
By proceeding as in Section 7.1 we finally get the equation (7.1.10).

7.3 Proper Lagrangian/Hamiltonian and Equations of Motion

In the foregoing analysis we have used the local time interval dt, in defining velocity, acceleration etc. On the other hand, Minkowski defined the velocity vector as the four vector $\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{cdt}{d\tau}\right)$ where $d\tau$ is the proper time. Accordingly, we define the proper Lagrangian/Hamiltonian by multiplying the classical Lagrangian/Hamiltonian with $\left(\frac{dt}{d\tau}\right)^2 = \left(\frac{dx^4}{cd\tau}\right)^2$ where we let $ct = x^4$; further we wrote x^1, x^2 and x^3 in place of x, y, z respectively. The classical Lagrangian/Hamiltonian is

(i) $\mathrm{L} = (\boldsymbol{p} - m\mathbf{A}_1).\boldsymbol{v} - (mc^2 - m\phi_1)$ (7.3.1)

where $m = \frac{m_0}{\sqrt{|1-v^2/c^2|}}$ (relativistic case)

(ii) $\mathrm{L} = (\boldsymbol{p} - m\mathbf{A}_1).\boldsymbol{v} - (mc^2 - m\phi_1)$ (7.3.2)

where $m = m_0 \,\mathrm{Exp}\left(\frac{v^2}{2c^2}\right)$ (non-relativistic case)

By multiplying each of these by $\left(\frac{dx^4}{cd\tau}\right)^2$ we have,

$$L_p = [(\boldsymbol{p} - m\mathbf{A}_1).\boldsymbol{v} - (mc^2 - m\phi_1)]\left(\frac{dx^4}{cd\tau}\right)^2$$

$$= \left[\left(m\frac{d\boldsymbol{r}}{dt} - m\mathbf{A}_1\right)\cdot\frac{d\boldsymbol{r}}{dt} - (mc^2 - m\phi_1)\right]\left(\frac{dx^4}{cd\tau}\right)^2$$

$$i.e. = \frac{md\boldsymbol{r}}{d\tau}\cdot\frac{d\boldsymbol{r}}{d\tau} - \left(m\frac{\mathbf{A}_1}{c}\right)\cdot\frac{d\boldsymbol{r}}{d\tau}\frac{dx^4}{d\tau} - \left(m - \frac{m\phi_1}{c^2}\right)\left(\frac{dx^4}{d\tau}\right)^2 \qquad (7.3.3)$$

which can be rewritten in the form

$$L_P = g_{ij}\ \dot{x}^i\,\dot{x}^j \qquad (7.3.4)$$

where dot denotes time rate w.r.t proper time. Hence Hamilton's principle of stationary action for

$$L_P \text{ is } \delta S = 0 \qquad (7.3.5)$$

where

$$S = \int L_p d\tau = \int g_{ij}\dot{x}^i\dot{x}^j d\tau \qquad (7.3.6)$$

where

$$g_{11} = g_{22} = g_{33} = m$$

$$g_{41} = \frac{-mA_{11}}{2c} = g_{14};\ g_{42} = \frac{-mA_{12}}{2c} = g_{24}\ ; g_{43} = \frac{-mA_{13}}{2c} = g_{34}$$

$$g_{44} = -(m - m\phi_1 c^{-2}) \text{ and } \mathbf{A}_1 = (A_{11}, A_{12}, A_{13})$$

We have noted earlier that the potentials are part of mass. Hence $\mathbf{g}_{ij}$ may be considered as potentials or curvatures of the external field as well as of the moving test particle consisting of gravitational and/or electromagnetic parts.

The Euler-Lagrange equation corresponding to (7.3.5) and (7.3.6) is

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} = 0$$

or

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^j}\right) - \frac{\partial L}{\partial x^j} = 0 \qquad (7.3.7)$$

By solving this equation, we get the Lorentz Force Law. In the former case (i) the Euler-Lagrange equation gives

$$\frac{-d\boldsymbol{p}}{d\tau} = m\mathbf{E}_1 + m\boldsymbol{v} \times B_1 \qquad (7.3.8)$$

and in the latter case (ii) we get

$$-\left[\left(1 + \frac{v^2}{c^2}\right)\frac{d\boldsymbol{p}}{d\tau} + \frac{m\boldsymbol{v} \times (\boldsymbol{v} \times \dot{\boldsymbol{v}})}{c^2} + \frac{mv^2}{c^2}\boldsymbol{v}\right]$$

$$= -m\left(\nabla\phi_1 + \frac{\partial \mathbf{A}_1}{d\tau}\right) + m\boldsymbol{v}\times(\nabla\times\mathbf{A}_1) \qquad (7.3.9)$$

where $\boldsymbol{v} = \frac{d\boldsymbol{r}}{d\tau}, \boldsymbol{p} = m\frac{d\boldsymbol{r}}{d\tau}, \dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{d\tau}, \mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial\tau}, \mathbf{B}_1 = \nabla\times\mathbf{A}_1$

Comparing $\mathrm{L_p} = g_{ij}\dot{x}^i\dot{x}^j$ (7.3.4) with the Minkowski metric for proper time

$$ds^2 = c^2 d\tau^2 = \mathrm{G}_{ij}\, dx^i\, dx^j \qquad (7.3.10)$$

we see that (7.3.4) and (7.3.10) are equivalent, except for their dimensions; $\mathrm{L_p}$ has the dimension of energy, whereas ds has the dimension of length.

For a general holonomic system let

$$\mathrm{L} = \mathrm{L}\left(x^1, x^2, \cdots x^n, t, \frac{dx^1}{dt}, \frac{dx^2}{dt} \cdots \frac{dx^n}{dt}\right) \qquad (7.3.11)$$

and $x^{n+1} = ct$

$$\therefore \mathrm{L} = \mathrm{L}\left(x^1, x^2, \cdots x^n, c^{-1}x^{n+1} + \frac{c\dot{x}^1}{\dot{x}^{n+1}}, \cdots \frac{c\dot{x}^n}{\dot{x}^{n+1}}\right)$$

where dot denotes time rate with respect to the proper time τ

$$\therefore \mathrm{L_p} = (\dot{x}^{n+1})^2 \mathrm{L}\left(x^1, x^2, \cdots x^n, c^{-1}x^{n+1}, \frac{c\dot{x}^1}{\dot{x}^{n+1}}, \cdots \frac{c\dot{x}^n}{\dot{x}^{n+1}}\right)$$

$$\text{i.e. } \mathrm{L_p} = \mathrm{L_p}\left(x^1, x^2, \cdots x^n, x^{n+1}, \dot{x}^1, \dot{x}^2, \cdots \dot{x}^{n+1}\right) \qquad (7.3.12)$$

$$\therefore \delta\mathrm{L_p} = \sum_1^{n+1}\frac{\partial \mathrm{L_p}}{\partial x^k}\delta x^k + \sum_1^{n+1}\frac{\partial \mathrm{L_p}}{\partial \dot{x}^k}\delta\dot{x}^k = \sum_1^{n+1}\frac{\partial \mathrm{L_p}}{\partial x^k}\delta x^k + \delta\sum_1^{n+1}\frac{\partial \mathrm{L_p}}{\partial \dot{x}^k}\delta\dot{x}^k - \sum_1^{n+1}\frac{d}{d\tau}\left(\frac{\partial \mathrm{L_p}}{\partial \dot{x}^k}\right)\delta x^k$$

$$i.e\ \delta\left[\sum \frac{\partial \mathrm{L_p}}{\partial \dot{x}^k}\dot{x}^k - \mathrm{L_p}\right] = \sum_1^{n+1}\frac{d}{d\tau}\left[\frac{\partial \mathrm{L_p}}{\partial \dot{x}^k} - \frac{\partial \mathrm{L_p}}{\partial x^k}\right]\delta x^k = 0 \qquad (7.3.13)$$

By using Lagrange's equations, we have

$$\frac{d}{d\tau}\left(\frac{\partial \mathrm{L_p}}{\partial \dot{x}^k}\right) - \frac{\partial \mathrm{L_p}}{\partial x^k} = 0 \quad \left(k = 1,2,\cdots(n+1)\right) \qquad (7.3.14)$$

By defining

$$H_P = \sum_{1}^{n+1} \left(\frac{\partial L}{\partial \dot{x}^k}\right) \dot{x}^k - L_P \quad (7.3.15)$$

as the proper Hamiltonian, we have from (7.3.13)

$$\delta H_P = 0 \quad (7.3.16)$$

∴ H_P is a constant independent of τ but involving t.

8. A Comparison of Maxwell-Lorentz Equations for Mass Particle and Electron

We have derived the equations for mass particle; in the same manner, we can derive the equations for electrons. We have omitted the derivations of these equations, as they are proved in books on electro-magnetism.

Description of Law	Gravito-rotational/Gravito-magnetic fields	Electro-magnetic fields
Gauss Law	$\text{Div}\, \mathbf{D}_1 = \rho_1^-$	$\text{Div}\, \mathbf{D}_2 = \rho_2$
Faraday's Law	$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}$	$\nabla \times \mathbf{E}_2 = -\frac{\partial \mathbf{B}_2}{\partial t}$
Field Intensity	$\mathbf{E}_1 = -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial t}$	$\mathbf{E}_2 = -\nabla\phi_2 - \frac{\partial \mathbf{A}_2}{\partial t}$
Flux Density	$\mathbf{B}_1 = \nabla \times \mathbf{A}_1$	$\mathbf{B}_2 = \nabla \times \mathbf{A}_2$
Equation of Continuity	$\frac{\partial \phi_1}{\partial t} + c^2 \nabla \cdot \mathbf{A}_1 = 0$	$\frac{\partial \phi_2}{dt} + c^2 \nabla \cdot \mathbf{A}_2 = 0$
Ampere's Law	$\nabla \times \mathbf{H}_1 = \mathbf{J}_1 + \frac{\partial \mathbf{D}_1}{\partial t}$	$\nabla \times \mathbf{H}_2 = \mathbf{J}_2 + \frac{\partial \mathbf{D}_2}{\partial t}$
Wave Equation of four potentials	$\nabla^2\phi_1 - \frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t^2} = \frac{\rho_1}{\varepsilon_1}$ $\nabla^2\mathbf{A}_1 - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_1}{\partial t^2} = \mu_1\mathbf{J}_1$	$\nabla^2\phi_2 - \frac{1}{c^2}\frac{\partial^2\phi_2}{\partial t^2} = \frac{-\rho_2}{\varepsilon_2}$ $\nabla^2\mathbf{A}_2 - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_2}{\partial t^2} = -\mu_2\mathbf{J}_2$
Energy Law	$u = \frac{1}{2}\int\left(\varepsilon_1 \mathbf{E}_1 \cdot \mathbf{E}_1 + \frac{1}{\mu_1}\mathbf{B}_1 \cdot \mathbf{B}_1\right) dV$	$u = \frac{1}{2}\int\left(\varepsilon_2 \mathbf{E}_2 \cdot \mathbf{E}_2 + \frac{1}{\mu_2}\mathbf{B}_2 \cdot \mathbf{B}_2\right) dV$
Lorentz' Force Law	$\mathbf{F}_1 = m^-(\mathbf{E}_1 + \boldsymbol{v} \times \mathbf{B}_1)$	$\mathbf{F}_2 = q(\mathbf{E}_2 + \boldsymbol{v} \times \mathbf{B}_2)$

9 Conclusions

The presence of four-potentials in L_P of the modified Newton/Lorentz Minkowski theory, and its absence in ds^2 of GTR makes them different theories. L_P has four-potentials as the different g_{ij} whereas in GTR, G_{ij} are arbitrary and G_{44} is chosen so that Newtonian scalar potential is obtained as an approximation. The

Ampere's vector potential is completely ignored in relativity. GTR stipulates that g_{ij} are connected with gravitation only and not connected with electro-magnetic effects or any other four-potentials. But the modified Newton-Lorentz theory includes both gravitation and electro magnetism in a single theory. GTR is a theory of gravitation based on the equations $\delta(ds) = 0$, $R_{ij} = 0$ where R_{ij} is the Ricci tensor and ds is the four-dimensional metric. This theory cannot lead to the Newton-Lorentz force law without extra assumption. There is no basis in assuming that non-constant g_{ij} is due to the presence of gravitation, and constant g_{ij} is due to its absence. The GTR excludes the possibility of a unified theory for the motion of mass particles and charged particles by highlighting the observation that the energy momentum tensor of EM field has a vanishing trace. On the other hand, the vector fields can be generalized by means of contracted tensor fields.

APPENDIX

MATHEMATICAL IDENTITIES

1. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
2. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$
3. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
5. **Flux Circulation Theorem (STOKES)**

Let **F** be a vector function of space co-ordinates, continuous with its first and second order partial derivative within and on the boundary C of an open surface S. Then the circulation of **F** over C is equal to the flux of $\nabla \times \mathbf{F}$ over S i.e.

$$\oint_C \mathbf{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \boldsymbol{n} dS$$

6. **Flux Divergence Theorem (GAUSS-OSTROGRADSKY)**

Let **F** be a vector function of space co-ordinates with continuous partial derivatives up to second order, within a close surface S enclosing a volume V. Then the flux of **F** over S is equal to the volume integral of Div **F** over V

$$i.e. \oiint_S \mathbf{F} \cdot \boldsymbol{n} dS = \iiint_V (\nabla \cdot \mathbf{F}) dV$$

7. **Helmholtz Theorem**

Let **F** be a vector function of space co-ordinates such that the circulation density $\mathbf{J} = \nabla \times \mathbf{F}$ and the source-density $\rho = \nabla \cdot \mathbf{F}$; then $\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}$ where

$$\phi = \frac{1}{4\pi} \iiint_V \frac{\rho dV}{R}, \mathbf{A} = \frac{1}{4\pi} \iiint_V \frac{\mathbf{J}\, dV}{R}, R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

is the distance of the field point $P(x, y, z)$ from the source points (x', y', z') of V and $dV = dx'dy'dz'$

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