Research Article
Open Access Journal of Physics & Mathematics
Received: 27-03-2025 **Accepted:** 01-04-2025 **Published:** 10-04-2025
OPENPUB GLOBAL PUBLISHERS
Volume 1 | Issue 2

A Comparative Analysis of The Modified Newtonian Theory with The Relativistic Theory

Dr. Chandramohanan MR*

Former Professor of Mathematics, Narayanaguru College of Engineering Manjalummood, Tamil Nadu, India

***Corresponding author:** Chandramohanan MR, Former Professor of Mathematics, Narayanaguru College of Engineering, 41/2674 (1) Sreenilayam, Manacaud PO, Thiruvananthapuram, Kerala, India.

Citation: *Dr. Chandramohanan M.R (2025) A Comparative Analysis of The Modified Newtonian Theory with The Relativistic Theory. Open Access Journal of Physics & Mathematics. Research Article 1(2): 01-17.*

Abstract

In this article we do a comparative study of the modified Newtonian Theory with the Relativistic Theory of gravitation and arrive at the conclusion that the former is superior to the latter, since the classical Newtonian Theory [12,13] can be modified according to the theory of Electro-dynamics and some of the claims of Relativistic Theory of gravitation are explicable by the modified theory whereas some of the relativistic arguments are self-contradictory.

Keywords and Notations

Local time interval (dt);

proper time interval $(d\tau)$;

gravitational/electromagnetic field intensity $(\bar{E}_1/\bar{E}_2)$;

gravitational/electromagnetic permittivity (ϵ_1/ϵ_2);

gravitational constant $\left(G = \frac{1}{4\pi\,\epsilon_1}\right)$;

mass of a source mass/Sun (M);

scaled mass of Sun $(\bar{m})$;

reduced mass (μ);

gravitational/electromagnetic permeability (μ_1/μ_2) satisfying $\epsilon_1\mu_1 c^2 = 1$ and $\epsilon_2\mu_2 c^2 = 1$ where c is the maximum signal velocity in the gravitational/electromagnetic fields;

gravitational/electromagnetic potential (ϕ_1/ϕ_2);

proper time-interval $d\tau$ satisfies $c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2 - dR^2 - R^2d\theta^2 - R^2\sin^2\theta d\varphi^2$ (Minkowski's metric);

STR/GTR stands for special theory of relativity/general theory of relativity;

four-potentials of gravitational fields $(\phi_1, \bar{A}_1)$;

four-potentials of electromagnetic fields $(\phi_2, \bar{A}_2)$;

four-potentials of combinations of fields $(\phi_3, \bar{A}_3)$;

flow density/current density for gravitational particles ($\rho_1 \boldsymbol{v}$ or $\mathbf{J}_1$);

current density for electrons ($\rho_2 \boldsymbol{v}$ or $\mathbf{J}_2$);

classical Lagrangian (L);

proper Lagrangian $(\mathrm{L_p})$ defined by $\mathrm{L_p} = \mathrm{L}\ (dt/d\tau)^2$;

m^- stands for $(-m)$ and $\boldsymbol{E}^-$ stands for $(-\boldsymbol{E})$;

v_p stands for phase velocity;

v_g for group/particle velocity satisfying $v_p v_g = c^2$.

1. Introduction

In references [1-5] it was shown that the modified Newtonian dynamics and electrodynamics have similar nature and both satisfy the Maxwell-Lorentz Equation. Hence it is meaningful to compare the combined Newton-Lorentz theory with the Relativistic Theory. Wolfgang Pauli had remarked [18] that since electron theory is in agreement with the special theory of relativity, the latter cannot produce results which are not already contained in the Pre-relativistic Lorentz' electron theory and (i) the relativists could not prove that the primed and unprimed coordinates in the LT are equally inertial (ii) the LT is not an orthogonal transformation, (iii) the relativists could not disprove the existence of author [6-8,16] it is meaningful to omit the STR from our comparison with the modified Newtonian Theory. Further we have shown in [1,2,4,5] that the two frames involved in LT are not equally inertial but one of them must be a preferred frame. This forces us to use one local space-time coordinates (unprimed) and another proper space-time co-ordinates; the relationship between these was already found in [4] and stated below. It was shown that the relativistic time-concept is self-contradictory according to the postulate of constancy of speed of light and the concept of retarded/advanced time occurring in electrodynamics. Also we have shown [2,4,5] that the postulate of constancy of speed of light does not lead to the LT but to a different non-linear transformation. Hence it is meaningful to consider the use of the concepts of length contraction and time dilation along with Minkowski's metric as a part of the modified Newtonian Theory but not of STR [10] and it is sufficient to omit the STR from consideration and examine the merits of the modified Newtonian Theory with the gravitational theory of GTR. Further we have derived a transformation involving the local co-ordinates (ct, x) and the proper co-ordinates (c_τ, x_τ) in the form [4]:

(i) $c\tau = \sqrt{|c^2t^2 - x^2|^2},\ x_\tau = \frac{ctx}{\sqrt{c^2t^2-x^2}}$ (1.1)

(ii) $c^2t^2 = \frac{1}{2}c^2\tau^2\left[\sqrt{1+\frac{4x_\tau^2}{c^2\tau^2}}+1\right]$ (1.2)

$$x^2 = \frac{1}{2}c^2\tau^2\left[\sqrt{1+\frac{4x_\tau^2}{c^2\tau^2}}-1\right] \quad (1.3)$$

which can be generalized by taking (i) $y = y_\tau, z = z_\tau$ by changing x^2 into $r^2 = x^2 + y^2 + z^2$, x_τ^2 into $x_\tau^2 + y_\tau^2 + z_\tau^2$ and (ii) by replacing the co-ordinates by their differentials along with the Minkowski's metric for the proper time. The above analysis leads to the omission of the primed co-ordinates from future considerations.

2. Analysis and criticisms on the Newtonian Theory

The Newtonian system needs a basis for concepts such as velocity, acceleration, etc. that is, some framework, relative to which these concepts are well defined. It is indeed a matter of great difficulty to discover and to distinguish the true motions of particular bodies from the apparent, because the parts of the immovable space in which those motions are performed, do by no means come under the observation of our senses. Newton was aware of the difficulty of defining absolute space, as were Euler and other mathematicians. We are confronted with the question: How can one give a meaning to the concept of velocity, if you do not have a space to refer? Therefore, Newton introduced the concept of absolute space and absolute time. At the same time, Newton recognized clearly that only relative quantities could be directly measured. Unlike his relationist contemporaries Huygens, Leibnitz, etc. he was convinced that a scientifically useful notion of motion could not be based on relational quantities. Instead, he sought to demonstrate how absolute quantities could be deduced from relative observations. A good substitute for the universal time interval is given by the proper time interval $d\tau = \sqrt{1-\frac{v^2}{c^2}}dt$ or equivalently $dt \approx \sqrt{1+\frac{v'^2}{c^2}}d\tau$, where dash denotes derivative with respect to τ. Similarly we can use the proper length ds_0 which satisfies $ds = \sqrt{1-\frac{v^2}{c^2}}ds_0$ or equivalently $ds_0 = \sqrt{1+\frac{v'^2}{c^2}}\,ds$ as the substitute for absolute spatial distance. These relations are known as the time-dilation and length contraction, prevalent before the advent of relativity. It may be noted that v is restricted by the condition $|v| < c$ whereas v' is unrestricted. When v/c and v'/c are very small, we may use the local space-time co-ordinate as a good approximation to the absolute time co-ordinate or use the transformation given at the end of Section 1.

However, if motion is relative and everything in the world is in motion, as it is in the Cartesian philosophy, a question arises according to Newton: how can one ever set up a determinate theory of motion? We first examine various criticisms against the Newtonian theory. We have discussed the many body problems on Newtonian premises by using the centre of mass co-ordinates and the principle of least time [3]. This leads to a confirmation of Newton's inverse square law. Some of the proper time metrics of general relativity are examined in the light of the modified Newtonian-theory and the modified Minkowski's metric of equation (4.8) of Section 4. By examining the Compton-shift analysis, it was shown that the momentum p consists of a mechanical part $m\boldsymbol{v}$ and a quantum-mechanical/electromagnetic part $i\hbar\boldsymbol{k}$ so that $\boldsymbol{p} = m\boldsymbol{v} \pm i\hbar\boldsymbol{k}$ [3]. In this article an interpretation of Schrodinger's wave function ψ as the quantum-mechanical/electromagnetic energy density is examined.

The Newtonian law of gravitation was not well received even in Newton's own University of Cambridge [6]. The inverse square law in its elementary form implies elliptic orbits for planets. Scientists consider this as a serious defect of the theory. Many astronomers believed that the law was not reconcilable with the observed motions of heavenly bodies. They admitted that the planetary orbits are ellipses as an approximation. By the end of the seventeenth century, departure from elliptic motion was confirmed by astronomical observations. The inequalities were of two kinds. First, there were disturbances, which sighted themselves after a time, to have no cumulative effect, known as periodic inequalities. Secondly, there were departures that have a cumulative effect known as secular inequalities; the best known of them are in the orbits of Jupiter and Saturn. Newtonian law cannot be applied in its static form to have a satisfactory explanation. However, we have proved that the inverse square law is always valid, by the use of centre of mass coordinate system; then Newton's theory implies a non-elliptic orbit as in GTR [3-5]. Marquis de Laplace in 1784 was able to assert that the great inequality of Jupiter, is not a secular inequality, but an inequality of long period, in fact 929 years. Laplace showed that the mean motions of planets could not have secular accelerations because of their mutual attractions. After the triumphant conclusion of Laplace's research on the great inequality of Jupiter and Saturn, there is still the outstanding unresolved problem, which formed a serious challenge to the Newtonian theory, namely the secular acceleration of the mean motion of the moon. The inequality is not secular but periodic, though the period is immensely long, in fact of millions of years. In the following paragraphs (i) to (iv) we examine the criticisms against Newtonian theory described in the book '*Einstein Studies*-Vol. 6' [9].

(i) In 1710, in his book, *Principles of Human Knowledge*, Berkeley commented that in space, the motion of two globes with a common centre of mass cannot be conceived by the imagination. But if we suppose that the sky of the fixed stars is created, suddenly from the conception of the approach of the globes to the different parts of the sky, the motion will be conceived. Berkeley argued further that philosophers, who have a greater extent of thought and jester notions of the system of things, discover even the earth itself to be moved. There is no mathematical, quantitative justification in Berkeley's comments.

(ii) In 1882, in his book, *The Science of Mechanics*, Mach proposed two equations to support his dissent with Newtonian mechanics. He pointed out that one cannot conclude from the flattening of the earth due to its rotation about its axis, that absolute space exists; all that one can conclude is that the effect is associated with the rotation of the earth relative to the rest of the matter in the universe. Newton's experiments with the rotating pail of water, simply informs that the relative rotation of water with respect to the sides of the bucket/pail produces no noticeable centrifugal forces. The distant heavenly bodies have no influence on the acceleration, but they have on the velocity [9]. He claims: "try to fix Newton's bucket and rotate the heaven of fixed stars and then prove the absence of centrifugal process". John D Norton [9] retorted that the challenge of Mach is futile, as the two cases are just one case, described differently. It is well known that the centrifugal forces due to rotation of earth and gravitational mass of the earth cannot be separated experimentally and for this reason, tables of the acceleration due to gravity at various points on the earth's surface often include the contribution of the centrifugal force due to the rotation of the earth. According to the opinion of Einstein, these forces are called gravitational forces originating from the rotation of the distant masses relative to the rotating system of co-ordinates. But Newton used the words 'fictitious' forces instead of the words 'gravitational' forces. The choice between 'fictitious' and 'gravitational' is a matter of taste only and is irrelevant; besides the critics did not substantiate anything logically or quantitatively. On the other hand the verbal exercises about the rotating pail are just equivalent to interpretations of the second term in the Newton-Lorentz force law presented in [1-5], viz.,

$$\frac{dp}{dt} = m^{-}\mathbf{E}_1 + m^{-}v \times \mathbf{B}_1$$

(2.1)

In 1912, Einstein is said to have produced [9] the first definite result in favour of relativistic theory of gravitation in contrast to Newton's theory: a new type of 'force' analogous to electromagnetic induction. Further, he is said to have obtained the result that the presence of a mass shell of mass M increases the inertial mass m of a point mass at the centre to $m + (\mathrm{M}m\mathrm{G}/\mathrm{R}c^2)$ which is same as $= m[1 + (\phi_1/c^2)]$. We have shown, in [3,5] that this is always true. But the above assertion of Einstein is in contradiction with his own principle of equivalence which states that inertial mass and gravitational mass are equal. But Newton never treated them at par. If they are equal, how can the mass $m + \frac{\mathrm{M}m\mathrm{G}}{\mathrm{R}c^2}$ containing both (i) inertial part and (ii) gravitational part be in agreement with the principle of equivalence? According to the modified Newtonian dynamics, this question does not arise, as that principle is not necessary. Hence, the inference is that GTR is a theory of contradictory ideas, contradictory to the special theory as well.

(iii) In the book [9] it is described thus: Mach points out in his book referred above, that the distance between two bodies moving purely inertially, satisfies the differential equation

$$\frac{d^2r}{dt^2} = \frac{1}{r}\left[a^2 - \left(\frac{dr}{dt}\right)^2\right] \qquad (2.2)$$

where a is a constant. He continues by saying that the direction and velocity of a mass $\bar{\mu}$ in space remain constant and we may employ the expression: the mean acceleration of the mass $\bar{\mu}$ with respect to the masses $m_1, m_2, \dots$ at distances $r_1, r_2, \dots$ is zero, or

$$\frac{d^2}{dt^2}\left(\frac{\sum mr}{\sum m}\right) = 0 \qquad (2.3)$$

Equation (2.2) is a kinematical equation and equation (2.3) has the appearance of a dynamical equation. From the definition of weighted arithmetic mean, we know that $\frac{\sum mr}{\sum m} = \mathrm{R}'$ represents the average value of r, the various distances of the mass-bodies $m_1, m_2, \dots$ from the test mass $\bar{\mu}$. Without loss of generality we can assume that $\bar{\mu}$ is situated at the centre of mass of the system containing $m_1, m_2, \dots$. Since $\sum mr$ is a series of positive non-decreasing terms, it is convergent only if the number of terms is finite. Therefore, the number of mass-bodies of the relationist universe is finite and the centre of mass exists. Therefore, equation (2.3) implies $\frac{d^2\mathrm{R}'}{dt^2} = 0$ or $\mathrm{R}' = \mathrm{A}t + \mathrm{B}$ and hence $\mathrm{R} \to \infty$ as $t \to \infty$. Without the assistance of Newtonian principles, the conclusion is that the average distance of the mass-bodies from their centre of mass tends to infinity *i.e.* the outer mass bodies fall off to infinity one by one. By comparing R′ with the definition of the radius of gyration $\overline{\mathrm{K}}$ where $\overline{\mathrm{K}}^2 = \frac{\sum mr^2}{\sum m}$ we see that R′ in this discussion, is merely a crude form of the radius of gyration and hence (2.3) does not contain any new physical principle. The first equation can be rewritten as:

$$\frac{(d\dot{r}/dt)}{a^2-\dot{r}^2} = \frac{1}{r} \; i.e. \frac{2\dot{r}d\dot{r}}{a^2-\dot{r}^2} = \frac{2\dot{r}dt}{r} = \frac{2dr}{r} = \frac{d(r^2)}{r^2} \; i.e. \frac{d(r^2)}{r^2} - \frac{2\dot{r}d\dot{r}}{a^2-\dot{r}^2} = 0.$$

Integrating this we have

$$r^2(a^2-\dot{r}^2) = \mathrm{C}_1^2 \quad i.e. \quad a^2-\dot{r}^2 = \frac{\mathrm{C}_1^2}{r^2} \quad i.e. \quad \dot{r}^2 = a^2 - \frac{\mathrm{C}_1^2}{r^2} \quad i.e. \quad \frac{dr}{dt} = \frac{\sqrt{a^2r^2-\mathrm{C}_1^2}}{r}$$

$$i.e. \quad \frac{rdr}{\sqrt{a^2r^2-\mathrm{C}_1^2}} = dt \quad i.e. \quad \frac{rdr}{\sqrt{r^2-\mathrm{C}_1^2a^{-2}}} = adt.$$

Integrating again,

$$\sqrt{r^2-\mathrm{C}_1^2a^{-2}} = at+b \qquad \therefore r^2 = (\mathrm{C}_1^2/a^2) + (at+\mathrm{C}_2)^2$$

Therefore, $r \to \infty$ as $t \to \infty$, the same conclusion as in the case of equation (2.3). This shows that (2.3) which has the resemblance of a dynamical equation, is devoid of any physical or mathematical significance.

(iv) Mach's equations (2.2) and (2.3) are mathematically flawed, since they are scalar equations, whereas a vector equation with three components is needed to specify the motion of a particle. In the same book [9] J.B. Barbour made the following comments:

> 'we must consider those perennial bogeymen, the so-called manifestly anti-Machian solutions of general relativity, especially matter free space-times and above all empty Minkowski space. We can go a long way to exorcizing these bogeymen, if we hold fast to the following principle. Any solution of pure geometro-dynamics, is not to be analyzed as a matter free structure in which test particles have inertia or as a structure that has a disconcerting resemblance to Newton's absolute space and time. For us, as opposed to mathematicians "paid by their math department", to find Einsteinian solutions, that process can never end'.

In this context it is meaningful to examine the opinions of Einstein on Mach. Einstein was fascinated by the Machian arguments till about 1920 and hoped to incorporate Mach's ideas into his theory of gravitation. This hope was not realized in the end; there are still several anti-Machian solutions in general relativity [20]. Einstein had lost his enthusiasm in the principle he proposed as the Machian principle. In his autobiographical notes, Einstein [9] writes:

> 'Mach conjectures that, in a truly rational theory, inertia would have to depend upon the interaction of the masses, precisely as was true for Newton's other forces, a conception which for a long time I considered as in principle, the correct one. It presupposes implicitly, however that the basic theory should be of the general type of Newton's mechanics: masses and their interactions as the original concepts.'

It is clear that none of the relationists/relativists have attempted to introduce the centre of mass frame in their criticisms on Newtonian Theory. We have shown [3] that the inverse square law of Newtonian is valid relative to this frame.

3. Interpretation of Schrodinger's wave function ψ according to the modified Newtonian Theory

It may be noted from Section 4.2.1 of [5] that the four potentials $q(\phi_2, \mathbf{A}_2)$ is a measure of the energy momentum felt by a charge q in the four-field $(\phi_2, \mathbf{A}_2)$ of a source charge Q. Similarly, $m(\phi_1, \mathbf{A}_1)$ is a measure of the energy-momentum felt by a mass-particle m in the four-field $(\phi_1, \mathbf{A}_1)$ of mass M. From the discussion in Section 4.2 of [5] we can infer that the total momentum and energy of a charged particle consists of a (i) mechanical/inertial part (ii) quantum mechanical part/electromagnetic part.

For an electron in an atom, mc^2 and $q\phi_2$ represent a measure of the energy due to (i) inertial mass m and (ii) electrostatic potential of the nucleus. Let ρ denote mass density so that ρc^2 is energy density; therefore energy momentum density due to mass content of the moving electron in the orbit is $(\rho c^2, \rho \boldsymbol{v})$. If ρ_c is charge density of the electron then the energy momentum density due to charge of the moving electron is $\rho_c(\phi_2, \mathbf{A}_2) = (\rho_c\, \phi_2, \rho_c\, \mathbf{A}_2)$, with the notations of [1,3,5]. By the equation of continuity and the gauge condition

$$\frac{1}{c^2}\frac{\partial}{\partial t}(\rho c^2) + \operatorname{div}(\rho \boldsymbol{v}) = 0 \qquad (3.1)$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}(\rho_c \phi_2) + \operatorname{div}(\rho_c \mathbf{A}_2) = 0 \qquad (3.2)$$

Combining these equations

$$\frac{\partial}{\partial t}(\rho + ic^{-2}\rho_c\phi_2) + \operatorname{div}(\rho\boldsymbol{v} + i\rho_c\mathbf{A}_2) = 0 \qquad (3.3)(a)$$

$$i.e.\ \frac{\partial \rho^*}{\partial t} + \operatorname{div} \boldsymbol{p}^* = 0 \qquad (3.3)(b)$$

$$\text{where } \boldsymbol{p}^* = \rho\boldsymbol{v} + i\rho_c\mathbf{A}_2, \qquad (3.4)(a)$$

$$\text{and } \rho^* = \rho + ic^{-2}\rho_c\phi_2 = \rho_{mech} + i\rho_{qm} \qquad (3.4)(b)$$

are all densities.

By Newton's law of motion,

$$\frac{d\boldsymbol{p}^*}{dt} = -\nabla(\rho^* c^2) = -\nabla(\rho c^2 + i\,\rho_c\phi_2) \qquad (3.5)$$

Taking divergence,

$$\operatorname{div} \dot{\boldsymbol{p}}^{*} = -c^{2}\nabla^{2}(\rho^{*}) \tag{3.6}$$

By defining L* and H* as Lagrangian and Hamiltonian densities,

we have $\mathrm{L}^{*} = \boldsymbol{p}^{*}.\boldsymbol{v} - \mathrm{H}^{*}, dS^{*} = \mathrm{L}^{*}dt = \boldsymbol{p}^{*}.d\boldsymbol{r} - \mathrm{H}^{*}dt$

$$\therefore \nabla \mathrm{S}^{*} = \boldsymbol{p}^{*}\text{and } -\frac{\partial \mathrm{S}^{*}}{\partial t} = \mathrm{H}^{*} = \rho^{*}c^{2}$$

$$\therefore \nabla^{2}\mathrm{S}^{*} = \operatorname{div} \boldsymbol{p}^{*} = -\frac{\partial \rho^{*}}{\partial t} = -\frac{\partial}{\partial t}\left(-\frac{1}{c^{2}}\right)\left(\frac{\partial \mathrm{S}^{*}}{\partial t}\right) = \frac{1}{c^{2}}\frac{\partial^{2}\mathrm{S}^{*}}{\partial t^{2}}$$

$$\therefore \left(\nabla^{2} - \frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}\right)\mathrm{S}^{*} = 0$$

From (3.3)(b), by differentiating partially w.r.t. t

$$\frac{\partial^{2}\rho^{*}}{\partial t^{2}} + \operatorname{div}\left(\frac{\partial \boldsymbol{p}^{*}}{\partial t}\right) = 0 \tag{3.7}$$

$$i.e. \frac{\partial^{2}\rho^{*}}{\partial t^{2}} + \operatorname{div}\left[\left(\frac{d\boldsymbol{p}^{*}}{dt}\right) - (\boldsymbol{v}.\nabla)\boldsymbol{p}^{*}\right] = 0$$

$$i.e. \frac{\partial^{2}\rho^{*}}{\partial t^{2}} - c^{2}\nabla^{2}\rho^{*} = \operatorname{div}\left[(\boldsymbol{v}.\nabla)\boldsymbol{p}^{*}\right]$$

$$i.e. \frac{1}{c^{2}}\frac{\partial^{2}\rho^{*}}{\partial t^{2}} - \nabla^{2}\rho^{*} = c^{-2}\operatorname{div}[(\boldsymbol{v}.\nabla)\boldsymbol{p}^{*}] \tag{3.8}$$

Again from (3.5), $\frac{\partial \boldsymbol{p}^{*}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{p}^{*} = -c^{2}\nabla\rho^{*}$

$$\therefore \frac{\partial^{2}\boldsymbol{p}^{*}}{\partial t^{2}} + \frac{\partial}{\partial t}[(\boldsymbol{v}.\nabla)\boldsymbol{p}^{*}] = -c^{2}\nabla\left(\frac{\partial\rho^{*}}{\partial t}\right) = c^{2}\nabla(\nabla\cdot\boldsymbol{p}^{*})$$

$$= c^{2}[\nabla^{2}\boldsymbol{p}^{*} + \nabla\times(\nabla\times\boldsymbol{p}^{*})] \tag{3.9}$$

$$i.e. \frac{\partial^{2}\boldsymbol{p}^{*}}{\partial t^{2}} - c^{2}\nabla^{2}\boldsymbol{p}^{*} = -\frac{\partial}{\partial t}[(\boldsymbol{v}.\nabla)\boldsymbol{p}^{*}] + c^{2}\nabla\times(\nabla\times\boldsymbol{p}^{*})$$

$$i.e. \frac{1}{c^{2}}\left(\frac{\partial^{2}\boldsymbol{p}^{*}}{\partial t^{2}}\right) - \nabla^{2}\boldsymbol{p}^{*} = -\frac{1}{c^{2}}\frac{\partial}{\partial t}[(\boldsymbol{v}.\nabla)\boldsymbol{p}^{*}] + \nabla\times(\nabla\times\boldsymbol{p}^{*}) \tag{3.10}$$

As an approximation, the RHS of (3.9) and (3.10) are negligible. Therefore, (3.9) and (3.10) become the wave equations.

$$\frac{1}{c^2}\frac{\partial^2(\rho^* c^2)}{\partial t^2} - \nabla^2(\rho^* c^2) \approx 0 \qquad (3.11)$$

where $\rho^* = \rho + ic^{-2}\,\rho_c\phi_2 = \rho_{mech} + i\rho_{qm}$ or $\rho^* c^2 = \rho^*_{mech}\,c^2 + i\rho^*_{qm}\,c^2$

$$\text{and}\,\frac{1}{c^2}\,\frac{\partial^2 \boldsymbol{p}^*}{\partial t^2} - \nabla^2 \boldsymbol{p}^* \approx 0 \qquad (3.12)$$

Both real and imaginary parts of (3.11) and (3.12) satisfy the homogeneous wave equation. To solve (3.11) by the method of separation of variables, we substitute

$$\rho^* c^2 = \psi\,(x,y,z)\mathrm{T}\,(t) \qquad (3.13)$$

where $\psi\,(x,y,z) = \psi_1\,(x,y,z) + i\,\psi_2\,(x,y,z)$, into (3.11). Taking imaginary parts we get

$$\frac{1}{c^2}\,\psi_2(x,y,z)\,\mathrm{T}'' = (\nabla^2\psi_2)\mathrm{T}$$

$$i.e.\ \frac{\nabla^2\psi_2}{\psi_2} = \frac{\mathrm{T}''}{c^2\mathrm{T}} \qquad \text{Let each} = -k^2$$

$$\therefore\ (\nabla^2 + k^2)\psi_2 = 0 \qquad (3.14)$$

$$\text{and } \mathrm{T}'' + k^2c^2\mathrm{T} = 0 \qquad (3.15)$$

The last equation has solutions $\mathrm{T} = \mathrm{A}\,\mathrm{Exp}(\pm\,ikct)$ and then $|\mathrm{T}| = \mathrm{A}$

$$\therefore\ \rho_{qm}c^2 = \psi_2(x,y,z)\mathrm{A}\,\mathrm{Exp}(\pm\,ikct) \qquad (3.16)(a)$$

$$\text{and } |\rho_{qm}c^2| = \mathrm{A}|\psi_2\,(x,y,z)| \qquad (3.16)(b)$$

$\therefore\ \psi_2\,(x,y,z)$ has the dimension of $\rho_{qm}c^2$

Therefore, the wave function $\psi_2(x,y,z)$ represents quantum mechanical energy density. From classical dynamics, we have T + V= constant = E where T is the kinetic energy and V is potential energy. Take $\mathrm{V}(r)$ as the potential energy per unit charge at the electron due to the nucleus of the atom. By using the mass velocity relation $m = m_0\mathrm{Exp}\left(\frac{v^2}{2c^2}\right)$ and substituting $\mathrm{T} = (m - m_0)c^2$, the energy equation gives

$$m_0c^2\mathrm{Exp}\left(\frac{v^2}{2c^2}\right) - m_0c^2 + q\mathrm{V}(r) = \mathrm{E} \qquad (3.17)$$

$$\therefore \text{Exp}\left(\frac{v^2}{2c^2}\right) = 1 + \frac{\mathrm{E} - qV(r)}{m_0c^2}$$

$$\therefore \boldsymbol{p}^2 = m_0^2\boldsymbol{v}^2\text{Exp}\,(v^2/c^2) = m_0^2 \cdot 2c^2 \log\left[1 + \frac{\mathrm{E} - qV(r)}{m_0c^2}\right]\left[1 + \frac{\mathrm{E} - qV(r)}{m_0c^2}\right]^2$$

$$= 2m_0^2c^2\left[\frac{\mathrm{E} - qV(r)}{m_0c^2} - \frac{1}{2}\left(\frac{\mathrm{E} - qV(r)}{m_0c^2}\right)^2 + \cdots\right]\left[1 + \left(\frac{\mathrm{E} - qV(r)}{m_0c^2}\right)\right]^2$$

Letting $\boldsymbol{p} = \hbar\boldsymbol{k}$ implies

$$\hbar^2\boldsymbol{k}^2 = 2m_0\big(\mathrm{E} - qV(r)\big)\left[1 - \frac{\mathrm{E} - qV(r)}{2m_0c^2} + \cdots\right]\left[1 + \frac{\mathrm{E} - qV(r)}{m_0c^2}\right]^2$$

$$\therefore \boldsymbol{k}^2 = \frac{2m_0}{\hbar^2}[\mathrm{E} - qV(r)]\left[1 + \frac{3}{2}\left(\frac{\mathrm{E} - qV(r)}{m_0c^2}\right) + \cdots\right] \qquad (3.18)$$

By using this approximation in equation (3.14) we get

$$\nabla^2\psi_2 + \frac{2m_0}{\hbar^2}[\mathrm{E} - qV(r)]\left[1 + \frac{3}{2}\left(\frac{\mathrm{E} - qV(r)}{m_0c^2}\right) + \cdots\right]\psi_2 = 0 \qquad (3.19)$$

By omitting second and higher powers of, $\mathrm{E} - qV(r)$ we get the Schrodinger's equation

$$\nabla^2\psi_2 + \frac{2m_0}{\hbar^2}[\mathrm{E} - qV(r)]\psi_2 = 0 \qquad (3.20)$$

Thus, the Schrodinger's wave function ψ_2 satisfying (3.14) and (3.20) has the dimension of quantum mechanical energy density. Substituting the solution of Schrodinger's equation (3.20) in to (3.16)(a), we get the imaginary part of the solution of equation (3.11). Similarly, we can find the real part of the solution of equation (3.11) and can show that ψ_1 is similar to ψ_2.

4. Adaptations from the Minkowski's metric

Adaptations from the Minkowski's metric [15] into (i) Schwarzschild form (ii) Robertson-Walker/Friedman form (iii) Eddington-Robertson form etc.

Books on GTR have descriptions of Schwarzschild metric, in the presence of a central field of a mass M, given by

$$c^2 d\tau^2 = [1-(2\mathrm{MG}/\mathrm{R}c^2)]c^2dt^2 - \left[\left(1-(2\mathrm{MG}/\mathrm{R}c^2)\right)\right]^{-1} d\mathrm{R}^2 - \mathrm{R}^2 d\theta^2 - \mathrm{R}^2 \sin^2\theta d\varphi^2$$

which is an adaptation of Minkowski's metric by changing the coefficients of dt^2 and dR^2 according to the wish of the user.

The co-ordinate singularity at the horizon of the Schwarzschild solution of the Schwarzschild metric given above has stimulated work towards a general method of determining whether a true singularity exists and there is not even a fully satisfactory definition of a true singularity [19,20]. We shall examine the truth of the statement of Bernard F. Schutz and the statement of Weinberg -*The discussion of the Schwarzschild singularity does not apply to any gravitational field actually known to exist anywhere in the universe. However, like Aesop's fables, it is useful....* [20]

By taking $c = 1$ we get the simple form

$$d\tau^2 = [1-(2\mathrm{MG}/\mathrm{R})]dt^2 - \left[\left(1-(2\mathrm{MG}/\mathrm{R})\right)\right]^{-1} d\mathrm{R}^2 - \mathrm{R}^2 d\theta^2 - \mathrm{R}^2 \sin^2\theta d\varphi^2 \qquad (4.1)$$

When $\mathrm{R} = 2\mathrm{MG}$, the coefficient of $d\mathrm{R}^2$ has a singularity. To get rid of this singularity, the usual exercise in GTR is to make the transformation

$$\mathrm{R} = r\left(1+\frac{\mathrm{MG}}{2r}\right)^2 \qquad (4.2)$$

or equivalently

$$r = \frac{1}{2}[\mathrm{R} - \mathrm{MG} + (\mathrm{R}^2 - 2\mathrm{MGR})^{½})] \qquad (4.3)$$

In addition, the Robertson-Walker/Friedman metric is stated to be

$$d\tau^2 = \left(1-\frac{2\mathrm{MG}}{\mathrm{R}}\right)dt^2 - \phi(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right] \qquad (4.4)$$

where $k = 0$ or ± 1

In this section, we examine these metrics, in the light of the modified Newton-Lorentz Theory. In the absence of gravitational/electromagnetic fields, the Minkowski's metric of proper time, is

$$d\tau^2 = dt^2 - d\mathrm{R}^2 - \mathrm{R}^2\,(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (4.5)$$

From the derivation of Planetary motion [1,5] we have

$$\mathrm{R}^2 \frac{d\theta}{dt} = h_0 \,\mathrm{Exp}\left(\frac{-\mathrm{MG}}{\mathrm{R}}\right)$$

By defining $\mathrm{R}^2 \frac{d\theta}{d\tau} = h_0$ we may write

$$\frac{\partial\tau}{\partial t} = \text{Exp}\left(\frac{-\text{MG}}{\text{R}}\right) \qquad (4.6)$$

Also from the equations for length contraction and time-dilation, we have [1,4,5]

$$dsdt = ds_0 d\tau \text{ or } d\,\text{R}\,dt \approx d\text{R}_0 d\tau$$

Hence $\frac{\partial \text{R}_0}{\partial R} = \frac{\partial t}{\partial \tau} = \text{Exp}\left(\frac{\text{MG}}{\text{R}}\right)$.

When dt becomes $\text{Exp}\left(\frac{-\text{MG}}{\text{R}}\right)dt$, $d\text{R}$ becomes $\text{Exp}\left(\frac{\text{MG}}{\text{R}}\right)d\text{R}$, so that the product is $dRdt$, therefore $\frac{\partial\tau}{\partial R} = \frac{\partial\tau}{\partial R_0}\frac{\partial \text{R}_0}{\partial R} = \frac{\partial\tau}{\partial R_0}\,\text{Exp}\left(\frac{\text{MG}}{\text{R}}\right)$ where $\frac{\partial\tau}{\partial R_0}$ being the ratio of proper values is assumed to be a constant, independent of local co-ordinates. Letting this constant to be unity, we write

$$\therefore \frac{\partial\tau}{\partial \text{R}} = \text{Exp}\left(\frac{\text{MG}}{\text{R}}\right) \qquad (4.7)$$

∴ In the central field of a mass M, the radial part $d\text{R}$ as well as the local time dt are affected. Accordingly, we can write Minkowski's metric in the presence of a centrally symmetric gravitational field of mass M by using Newtonian premises in the form

$$d\tau^2 = \text{Exp}\left(\frac{-2\text{MG}}{\text{R}}\right)dt^2 - \text{Exp}\left(\frac{2\text{MG}}{\text{R}}\right)d\text{R}^2 - \text{R}^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (4.8)$$

This metric derived according to Newtonian Theory has no singularity at $\text{R} = 2\text{MG}$ or any other positive value. On the other hand, the metric (4.8) can be manipulated to the Schwarzschild metric (4.1) by using the approximations

$$\text{Exp}\left(\frac{-2\text{MG}}{\text{R}}\right) \approx 1 - \frac{2\text{MG}}{\text{R}} \text{ and } \text{Exp}\left(\frac{2\text{MG}}{\text{R}}\right) = \frac{1}{\text{Exp}\left(\frac{-2\text{MG}}{\text{R}}\right)} \approx \left(1 - \frac{2\text{MG}}{\text{R}}\right)^{-1}$$

Similarly the Eddington-Robertson form of the metric can be obtained by writing $\text{Exp}\left(\frac{-2\text{MG}}{\text{R}}\right) \approx 1 - \frac{2\text{MG}}{\text{R}} + \frac{2\text{M}^2\text{G}^2}{\text{R}^2}$.

Since equation (4.8) has no singularity at $\text{R} = 2\text{MG}$, the conclusions of GTR based on the metrics (4.1) and/or (4.4) or any such metrics, are invalid according to the modified Newton-Lorentz theory based on equation (4.8). Next we shall derive the transformations (4.2)/(4.3) from (4.8) of the modified Newton-Lorentz theory. We can rewrite equation (4.8) in the isotropic form

$$d\tau^2 = \text{Exp}\left(\frac{-2\text{MG}}{\text{R}(r)}\right)dt^2 - \text{Exp}[2\lambda(r)](dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2) \qquad (4.9)$$

by using the transformation

$$\mathrm{R} = r\ \mathrm{Exp}\ \lambda(r) \tag{4.10)(a}$$

$$\mathrm{Exp}\left(\frac{\mathrm{MG}}{\mathrm{R}}\right) d\mathrm{R} = \mathrm{Exp}\ \lambda(r) dr \tag{4.10)(b}$$

Equation (4.10b) imply $\frac{dr}{r} = \mathrm{Exp}\left(\frac{\mathrm{MG}}{\mathrm{R}}\right)\frac{d\mathrm{R}}{\mathrm{R}}$

$$= \frac{1}{\mathrm{R}}\left(1 + \frac{\mathrm{MG}}{\mathrm{R}} + \frac{\mathrm{M}^2\mathrm{G}^2}{2\mathrm{R}^2} + \cdots\right) d\mathrm{R}$$

$$\text{Integrating} \quad r = \mathrm{R}\ \mathrm{Exp}\left(-\frac{\mathrm{MG}}{\mathrm{R}} - \frac{\mathrm{M}^2\mathrm{G}^2}{4\mathrm{R}^2} \cdots\right) \tag{4.11}$$

From (4.10a) and (4.11) we have

$$\lambda(r) = \frac{\mathrm{MG}}{\mathrm{R}} + \frac{\mathrm{M}^2\mathrm{G}^2}{4\mathrm{R}^2} + \cdots \tag{4.12}$$

Now R has to be eliminated by finding R as a function of r in the form $\mathrm{R} = \mathrm{R}(r)$

From (4.11) we have $r \approx \mathrm{R}$, for large values of r and R.

Using this initial approximation on the RHS of (4.11) we find the next approximation

$$r \approx \mathrm{R}\ \mathrm{Exp}\left(-\frac{\mathrm{MG}}{r} - \frac{\mathrm{M}^2\mathrm{G}^2}{4r^2}\right) \approx \mathrm{R}\left(1 - \frac{\mathrm{MG}}{r} - \frac{\mathrm{M}^2\mathrm{G}^2}{4r^2} + \frac{\mathrm{M}^2\mathrm{G}^2}{2r^2}\right)$$

$$i.e. \qquad r \approx \mathrm{R}\left(1 - \frac{\mathrm{MG}}{2r}\right)^2 \text{ or } \mathrm{R} \approx \frac{r}{\left(1 - \frac{\mathrm{MG}}{2r}\right)^2}$$

$$i.e. \qquad \mathrm{R} \approx r\left(1 + \frac{\mathrm{MG}}{2r}\right)^2 \tag{4.13}$$

This is the transformation (4.2) of GTR, from which we obtain (4.3) on inversion. Substituting the value of $\mathrm{R} = r\left(1 + \frac{\mathrm{MG}}{2r}\right)^2$ in (4.12) we have $\lambda(r)$ as a series of powers of r. By using these values in (4.9) we get the isotropic form without any singularity.

Next we shall find the transformation which transforms (4.8) into the Robertson-Walker form

$$d\tau^2 = \mathrm{Exp}\left(-\frac{2\mathrm{MG}}{\mathrm{R}(r)}\right)dt^2 - \mathrm{Exp}(2\mu(r))\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \quad (4.14)$$

Comparing (4.8) and (4.14) we have to define

$$\mathrm{R} = r\ \mathrm{Exp}\ \mu(r) \quad (4.15)(\mathrm{a})$$

and
$$\mathrm{Exp}\left(\frac{\mathrm{MG}}{\mathrm{R}}\right)d\mathrm{R} = \frac{\mathrm{Exp}\mu(r)dr}{\sqrt{1-kr^2}} \quad (4.15)(\mathrm{b})$$

From (4.15) we have $\mathrm{Exp}\left(\frac{\mathrm{MG}}{\mathrm{R}}\right)\frac{d\mathrm{R}}{\mathrm{R}} = \frac{dr}{r\sqrt{1-kr^2}}$

Integrating both sides

$$\int \frac{1}{\mathrm{R}}\left(1 + \frac{\mathrm{MG}}{\mathrm{R}} + \frac{\mathrm{M}^2\mathrm{G}^2}{2\mathrm{R}^2} + \cdots\right)d\mathrm{R} = \int \frac{dr}{r\sqrt{1-kr^2}}$$

$$\log\left[\mathrm{R}\ \mathrm{Exp}\left(-\frac{\mathrm{MG}}{\mathrm{R}} - \frac{\mathrm{M}^2\mathrm{G}^2}{4\mathrm{R}^2} - \cdots\right)\right] = \left\{\begin{array}{lll} \log r & \text{when} & k = 0 \\ \log\left|\frac{1-\sqrt{1-kr^2}}{r}\right| & \text{when} & k > 0 \\ -\log\left|\frac{1+\sqrt{1-kr^2}}{r}\right| & \text{when} & k < 0 \end{array}\right\}$$

except for the constant of integration

Similarly it is possible to find $\mu(r)$ by the method of finding $\lambda(r)$, as described above. Thus we get the Robertson-Walker form (4.14). In order to write (4.14) in the form of Friedman metric, we replace $\mathrm{Exp}\ [2\mu(r)]$ in it by an arbitrary factor $\phi(t)$ depending on the local time, as is done in general relativity. Reference [17] contains an exposition of another metric known as Kerr metric, but the author ends with the comment: 'we do not know what the source is for the Kerr metric or even whether such a metric can be realised in nature'. Thus, it is possible to get the metrics of general relativity, with or without any singularity. Further M.W. Evans has definitively refuted the Einsteinan General Relativity, emphasizing the fact that Einstein had not used the concept of tortion of the four-space [11].

From the above analysis, it is clear that the metrics in general relativity can be constructed by inflating or deflating the metric coefficients G_{ij} in the Newtonian form (4.8) of Minkowski's metric for proper time, according to the requirements of the user. Thus, general relativity prescribes singularities, which are derivable from Minkowskis's metric according to Newton-Lorentz theory.

In the modified Newtonian dynamics, we are justified to use the retarded-potentials and retarded fields relative to centre of mass frame [3]. On the other hand the Schwarzschild metric (4.1) has the special disadvantage that it has a singularity at $r = 2\mathrm{MG}$ which is non-existent according to Newton-Lorentz theory. Further, general relativity does not consider Ampere's vector potential and the centre of mass frames. Hence GTR cannot be considered as a generalization of modified Newton-Lorentz theory. There have been many attempts to imbed the theory of electromagnetism into the framework of an extended theory of general relativity by geometrization. However no unified theory of electromagnetism and gravitation had been developed which is convincing and satisfactory as the gravitational theory of general relativity. Weyl's theory of electromagnetism has a high degree of consistency and elegance but it has led to no prediction of any new physical phenomena, which could be observed as confirmation of Weyl's theory. On the contrary, in an appendix to Weyl's exposition of his unified field theory, Einstein raised some very serious objections to it on empirical physical grounds and concluded that Weyl's theory is in contradiction to experience.

These justify the comments of Bernard Schute and Steven Weinberg [19,20].

The discussion of the Schwarzschild singularity does not apply to any gravitational field actually known to exist anywhere in the universe. However, like Aesop's fables, it is useful.... [20]

It is clear that these relativistic arguments are based on the Newtonian form in equation (4.8) of the modified Minkowskis's metric, a part of the modified Newtonian Theory.

5. Conclusions

Reference [1] contains the deduction of Helmholtz' equation $(\nabla^2 + k^2)\phi = 0$ from Newtonian premises. The filiform solutions [6] of the Helmholtz' wave-equation $\nabla^2\psi - \frac{v^2}{c^4}\frac{\partial^2\psi}{\partial t^2} = 0$ are the null geodesics of the metric whose line-element satisfies $\frac{c^4}{v^2}dt^2 - dx^2 - dy^2 - dz^2 = 0$ and the null geodesics are curves which satisfy the Newtonian equation $\frac{md^2x}{dt^2} = \frac{-\partial V}{\partial x}$. Hence $\nabla^2 V - \frac{v^2}{c^4}\frac{\partial^2 V}{\partial t^2} = 0$ is the wave equation associated with the Newtonian equation of motion. Schrodinger's equation and de Broglie's equation/Klien-Gordon equation can be deduced from Helmoltz' equation which can be deduced from Newtonian premises [1].

Thus it is also possible to dispense with the primed co-ordinates in the LT and the postulates of STR can be replaced by the length contraction and time dilation in order to arrive at a transformation connecting the local space-time co-ordinates and the proper space-time co-ordinates [1]. The orbit of Planatery Motion is derivable from the modified Newton-Lorentz' Theory [1,3-5]. The LT represents actually the Doppler-shift and aberration formulae in disguise. This follows from the discussions in [1,2,4].

The many-body problem can be treated by using the concepts of Newton's inverse square law, centre of mass co-ordinates [3] and retarded potentials. Since the concept of vector potential is absent in GTR, whereas it is a part of the modified Newtonian dynamics and the Schwarzschild metric of GTR is a special case of the metric (4.8) of the modified Newtonian theory, the GTR cannot be considered as a generalized version of STR/modified Newtonian dynamics. Further GTR stipulates that there exists no preferred frame of reference.

But according to Newton-Lorentz Theory, the centre of mass frame is a preferred frame of reference in the study of many body problem. Further M.W. Ewans [11] has definitively refuted Einstein's general relativity which is devoid of the concept of torsion of space-time.

References

1. Chandramohanan MR (2024) About some applications of Mathematics, Journal of Physics & Optics Sciences 6.
2. Chandramohanan MR (2024) A Brief Account of the Concepts of Time in the Pre-Relativistic Electrodynamics and STR, Journal of Physics & Optics Sciences 6.
3. Chandramohanan MR (2024) About Newton's Law of Gravitation and the Energy Momentum Relations, Journal of Physics & Optics Sciences 6.
4. Chandramohanan MR (2024) Field Dependant Metric for Gravitational/EM Fields and a Non-Linear Transformation for Local to Proper Space-Time World, Journal of Physics & Optics Sciences 6.
5. Chandramohanan MR (2025) Maxwell-Lorentz Theory of Gravito-Rotational/-Magnetic Fields, Open Access Journal of Physics & Mathematics 1.
6. Whittaker ET (1960) A History of the Theories of Aether and Electricity, Nelson, London, Reprinted by Harper, New York.
7. Shtyrkov Eugene I (2005) Observation of Ether Drift in Experiments with Geostationary Satellites, Proceedings of the Natural Philosophy Alliance (NPA), STORRS, USA 2: 201-205.
8. Dingle H (1971) Science at the Crossroads, (Martin Brian & O'Keeffe, London, 1972); M. Sachs. Phys. Today 24-23.
9. Barbour JB, Pfister H (Eds.) (1995) Mach's principle: from Newton's bucket to quantum gravity 6. Springer Science & Business Media.
10. Einstein A (1946) The Principle of Relativity (2nd edition), Princeton.
11. Evans MW (2012) Definitive Refutations of the Einsteinian General Relativity. Cambridge Internat. Science Publ. https://www.vivabooksindia.com/book/definitive-refutations-of-the-einsteinian-general-relativity.
12. Feynman RP, Leighton RB etc. (1989). Feynman Lectures on physics, Addison-Wesley, New York 2.
13. Griffith David J (2009) Introduction to Electro Dynamics, PHI Learning Pvt. Ltd., New Delhi.
14. Lorentz HA (1992) Theory of Electrons, Dover Publications Inc., New York.
15. Lorentz HA, Einstein A, Minkowski H, Weyl H (1952) The Principle of Relativity, Dover Publications, New York.
16. Nordenson, Harold (1969). Relativity, Time and Reality, George Allen and Unwin Ltd, Ruskin House, London.
17. Padmanabhan T (2010) Gravitation: foundations and frontiers. Cambridge University Press.
18. Pauli Wolfgang (1958). Theory of Relativity, Pergamon Press, London.
19. Schute, Bernard F(2002) A First Course in General Relativity, Cambridge University Press, U.K.
20. Weinberg Steven (2008) Gravitation and Cosmology, Wiley Student Edition, India, New Delhi.

APPENDIX

MATHEMATICAL IDENTITIES

1. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
2. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$
3. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
5. Flux Circulation Theorem (STOKES)

Let **F** be a vector function of space co-ordinates, continuous with its first and second order partial derivative within and on the boundary C of an open surface S. Then the circulation of **F** over C is equal to the flux of $\nabla \times \mathbf{F}$ over S i.e.

$$\oint_{C} \mathbf{F} \cdot d\boldsymbol{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot \boldsymbol{n} dS$$

6. Flux Divergence Theorem (GAUSS-OSTROGRADSKY)

Let **F** be a vector function of space co-ordinates with continuous partial derivatives up to second order, within a close surface S enclosing a volume V. Then the flux of **F** over S is equal to the volume integral of Div **F** over V

$$i.e. \oiint_{S} \mathbf{F} \cdot \boldsymbol{n} dS = \iiint_{V} (\nabla \cdot \mathbf{F}) dV$$

7. *Helmholtz* Theorem

Let **F** be a vector function of space co-ordinates such that the circulation density **J**=∇×**F** and the source-density $\rho = \nabla \cdot \mathbf{F}$; then $\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}$ where

$$\phi = \frac{1}{4\pi} \iiint_{V} \frac{\rho dV}{R}, \mathbf{A} = \frac{1}{4\pi} \iiint_{V} \frac{\mathbf{J}\, dV}{R}, \ R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

is the distance of the field point $P(x, y, z)$ from the source points (x', y', z') of V and $dV = dx'dy'dz'$