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Ribbonness on boundary surface-link

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Abstract

It is shown that a boundary surface-link in the 4-sphere is a ribbon surface link if the surface-link obtained from it by surgery along a pairwise nontrivial fusion 1-handle system is a ribbon surface-link. As a corollary, the surface knot obtained from the anti-parallel surface-link of a non-ribbon surface-knot by surgery along a nontrivial fusion 1-handle is a non-ribbon surface-knot. This result answers Cochran's conjecture on non-ribbon sphere-knots in the affirmative.

Keywords: Boundary surface-link, Ribbon surface-link, Anti-parallel surface link, Cochran's conjecture.

Mathematics Subject Classification 2010: Primary 57N13; Secondary 57Q45

Introduction

A **surface-link** is a closed (possibly disconnected) closed oriented surface F smoothly embedded in the 4-sphere S^4. When F is connected, F is called a **surface-knot**. The components F_i (i =1, 2, ..., r) of F are 2-spheres, then F is called a **sphere-link** or an S^2-link of r components. A 1-handle system on a surface-link F is a 1-handle system h of disjoint 1-handles h_j (j = 1, 2,...,s) on F smoothly embedded in S^4. Let $F(h)$ be the surface-link obtained from F by surgery along a 1-handle system h. A **ribbon surface-link** is the surface-link $O(h)$ obtained from a trivial S2-link O by surgery along a 1-handle system h, [1], [2]. A surface-link F is a **boundary surface-link** if there is a system V of disjoint compact connected oriented 3-manifolds V_i (i =1, 2, ..., r) smoothly embedded in S^4 with $\partial V_i = F_i$ (i = 1, 2, ..., r). The system V is called a **disjoint Seifert hypersurface system** for the boundary surface-link F. The 1-handle system h on F is a **fusion 1-handle system** if $F(h)$ has the r −s (≥1) connected components, where the 1-handles h_j (j = 1, 2, ..., s) of h are called **fusion 1-handles**. Let F be a boundary surface-link in S^4 of surface-knot components F_i (i = 1, 2, ..., r), and V a disjoint Seifert hypersurface system for F consisting of V_i (i = 1, 2, ..., r) with $\partial V_i = F_i$ (i = 1, 2, ..., r). Let h be a fusion 1-handle system of fusion 1-handles h_j (j = 1, 2, ..., s) on F. Let D be a disk system of r disjoint disks D_i ($\subset F_i$) (i = 1, 2, ..., r), and B a system of disjoint 3-balls B_i (i = 1, 2, ...,r) in S^4 with $B_i \cap V_i = (\partial B_i) \cap V_i = D_i$ (i = 1, 2, ..., r). Since the 3-manifold $V_i \cup B_i$ is cell-move equivalent to V_i, the surface-link F is canonically equivalent to the surface-link $\cup^r_{i=1} \partial (V_i \cup B_i)$. The 3-ball system B and trivial S^2-link $L = \partial B$ are called a **local 3-ball system** and a **local S^2-link** of the boundary surface-link F, respectively. By sliding the transverse disk system of h meeting V into B along $V\cup B$, the fusion 1-handle system h on F is considered as a fusion 1-handle system h^L on the local S^2-link L, which is attached to the disk system $D' = \mathrm{cl}(\partial B \setminus D)$, so that h^L does not meet V, but transversely meets the interior of B with a transverse disk system. This fusion 1-handle system h^L is called a **localized fusion 1-handle system** of h on F. The surface-link $L(h^L)$ obtained from L by surgery along h^L is a ribbon S^2-link. A fusion 1-handle h^L_j of h^L is **trivial** if the core arc c^L_j of h^L_j is made disjoint from the interior of the 3-ball system B by a ∂-relative isotopy of c^L_j keeping L fixed. Otherwise, h^L_j is **nontrivial**. A fusion 1-handle h_j in h is **trivial** or **nontrivial** according to whether the corresponding fusion 1-handle h^L_j in h^L is trivial or nontrivial, respectively. The fusion 1-handle system h^L on L is **pairwise nontrivial** if every h^L_j in h^L is nontrivial and disjoint from the subsystem of B excluding two members attached by h^L_j. The pairwise nontrivial fusion 1-handle system h^L on L is deformable into a neighborhood of any arc system α of disjoint simple arcs α_j (j = 1, 2, ..., s) in S^4 such that $\partial\alpha_j = \partial c^L_j$ (j = 1, 2, ..., s) and $\alpha \cap B = \partial\alpha \cap \partial B = \partial\alpha$. This means that the pairwise nontrivial fusion 1-handle system h^L on the local S^2-link L is determined regardless of how the disjoint Seifert hypersurface system V for the boundary surface-link F is chosen. A fusion 1-handle system h on F is **pairwise nontrivial** if h^L on L is pairwise nontrivial.

The following theorem is a main result of this paper, correcting an incorrect result, [3, Theorem 1.4]. Some counterexamples are given, [4].

Theorem 1.1. Let F be a boundary surface-link of $r(\geq 2)$ components in S^4. If the surface-link $F(h)$ obtained from F by surgery along a pairwise nontrivial fusion 1-handle system h is a ribbon surface-link, then the surface-link F is a ribbon surface link with h belonging to the 1-handle system of the ribbon surface-link $F(h)$.

For a surface-knot F in S^4, let $F \times [0,1]$ be a normal $[0,1]$-bundle over F in S^4 such that the natural homomorphism $H_1(F \times 1;Z) \to H_1(S^4 \setminus F \times 0;Z)$ is the zero map. In other words, take $F \times [0,1]$ a boundary collar of a compact connected oriented 3-manifold V smoothly embedded in S^4 with $\partial V = F$, [2]. The surface-link $P(F) = \partial(F \times [0,1]) = F \times 0 \cup F \times 1$ in S^4 is called the *anti-parallel surface-link* of F, where by convention $F \times 0$ and $F \times 1$ are identified with $-F$ (i.e., the orientation reversed F) and F, respectively. The anti-parallel surface-link $P(F)$ is a boundary surface-link, because $P(F)$ is the boundary of $V \times 0 \cup V \times 1$ for a normal $[0,1]$-bundle $V \times [0,1]$ of a compact connected oriented 3-manifold V with $\partial V = F$ smoothly embedded in S^4. The half parti of the following theorem is a direct consequence of Theorem 1.1.

Theorem 1.2. Let $P(F)$ be the anti-parallel surface-link of a non-ribbon surface knot F in S^4, and $P(F)(h)$ the surface-knot obtained from $P(F)$ by surgery along a fusion 1-handle h. According to whether h is a trivial or nontrivial 1-handle, the surface-knot $P(F)(h)$ is a trivial or non-ribbon surface-knot, respectively.

Theorem 1.2 positively answers Cochran's conjecture on non-ribbon ability of the S^2-knot $P(F;h)$ for a non-ribbon S^2-knot F and any sufficiently complicated fusion1handle h, [1].

Proofs of Theorem 1.1 and 1.2

The proof of Theorem 1.1 is done as follows.

2.1: Proof of Theorem 1.1. If $F(h)$ is a disconnected ribbon surface-link, then there is a pairwise nontrivial fusion 1-handle system h^+ on F extending h such that $F(h^+)$ is a ribbon surface-knot. Thus, assume that $F(h)$ is a ribbon surface-knot. First, the proof of the case r=2 is given where h is a nontrivial fusion 1-handle on F. Let i=1 or 2. For the 3-manifold V_i with $\partial V_i = F_i$, let β_i be a 1-handle system on F_i embedded in V_i and disjoint from h and B such that $V_i' = \mathrm{cl}(V_i \setminus \beta_i)$ is a handlebody, [4]. Then the surface-link $F(\beta) = F_1(\beta_1) \cup F_2(\beta_2)$ for the 1-handle system $\beta = \beta_1 \cup \beta_2$ bounds the disjoint handlebody system $V' = V_1' \cup V_2'$ and hence is a trivial surface-link in S^4. The surface-knot $F(\beta)(h)$ is a ribbon surface-knot which is equivalent to the connected sum of the non-trivial ribbon S^2-knot $L(h)$ and the trivial surface-knots $F_i(\beta_i)(i = 1,2)$ attaching along the disks $D_i(i = 1,2)$. The ribbon S^2-knot $L(h)$ has a canonical SUPH system $W(Lh) = B^{(0)} \cup h$, where $B^{(0)} = B_1^{(0)} \cup B_2^{(0)}$ for a once-punctured 3-ball $B_i^{(0)}$ of the 3-ball B_i in the 3-ball system $B = B_1 \cup B_2$. Then the ribbon surface-knot $F(\beta)(h)$ has a SUPH system $W = W(Lh) \cup V'$ which is a disk sum of $W(Lh)$ and V' pasting along the disk system $D = D_1 \cup D_2$. On the other hand, the surface-knot $F(h)$ is a ribbon surface-knot and hence has a SUPH system $W(Fh)$. If necessary, by replacing $W(Fh)$ with a multi-punctured $W(Fh)$, the union $W' = W(Fh) \cup \beta$ is a SUPH system for the surface-knot $F(\beta)(h)$. By replacing W and W' with multi-punctured W and W', respectively, there is an orientation-preserving diffeomorphism f of S^4 sending W and W'. See Appendix of [9] for this meaning. In this paper, the following property is used.

(2.1.1) The diffeomorphism f of S4 is isotopically deformed so that the restriction of f to F(β)(h) is the identity map.

By (2.1.1), assume that the restriction $f|F(\beta)(h)$ is the identity. Let $D(h)$ be a transversal disk of the 1-handle h, and $D(\beta)$ a transversal disk system of the 1-handle

This means that $F = F_1 \cup F'$ is a ribbon surface-link with h belonging to the ribbon 1-handle system of the ribbon surface-knot $F(h)$. This completes the proof of Theorem 1.1 under the assumption of (2.1.1).

(2.1.1) is proved as follows.

Proof of (2.1.1). Let $F(\beta)(h)^\wedge$ be the ribbon S^2-knot obtained from $F(\beta)(h)$ by surgery along the O2-handle basis $(D_k, D_{k'})$ on $F(\beta)(h)$, which is isotopic to the ribbon S^2-knot $L(h)$ taking h as a localized single nontrivial pairwise fusion 1-handle system on the trivial S^2-link L, [7]. The image $f(F(\beta)(h)^\wedge)$ is a ribbon S^2-knot obtained from the ribbon surface-knot $f(F(\beta)(h))$ by surgery along the O2-handle basis $(f(D_k), f(D_{k'}))$, which is isotopic to the ribbon S^2-knot $f(L(h))$. Any S^2-knot K equivalent to L(h) is isotopic to L(h). In fact, K is written as a ribbon S^2-knot $L(h')$ for a fusion 1-handle h′ on the trivial S^2-link system $L = \partial B \cup \partial B_2$ with the same attaching part as h (since a trivial S^2-link is isotopically unique), and then the 1-handle h′ is deformed into the 1-handle h by an isotopy keeping the attaching part fixed. This is because equivalent ribbon S^2-knots L(h) and L(h′) are faithfully equivalent, [10]. Thus, the ribbon S^2-knot f(L(h)) is isotopic to L(h), meaning that $f(F(\beta)(h)^\wedge)$ is isotopic to $F(\beta)(h)^\wedge$. Then f is modified to have the property that the restriction of f to $F(\beta)(h)^\wedge$ is the identity. Further, there is a modification of f such that the restriction of f to $F(\beta)(h)$ is the identity map by uniqueness of an O2-handle pair in the soft sense, [4]. This completes the proof of (2.1.1).

This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is done as follows.

2.2: Proof of Theorem 1.2. Assume that the fusion 1-handle h on the anti-parallel surface-link P(F) is nontrivial. If the surface-knot P(F)(h) is a ribbon surface-knot, then the boundary surface link P(F) is a ribbon surface-link by Theorem 1.1, so that F is a ribbon surface-knot, contradicting that F is a non-ribbon surface-knot.

Thus, P(F)(h) is a non-ribbon surface-knot. Assume that h is a trivial fusion 1-handle on $P(F)=F_0 \cup F_1$ with $F_i = F \times i (i=0,1)$. Since P(F) is a boundary surface-link, there is a disconnected Seifert hypersurface $V_* = V_0 \cup V_1$ for P(F) with $\partial V_i = F_i (i=0,1)$ disjoint from the fusion 1-handle h except for the attaching part to P(F). By taking the original V_0 as V, it is possible to repace V_* with the union of $V_i = V \times i (i=0,1)$ such that h is disjoint from $V \times [0,1]$ except the attaching part to P(F). Let β be a 1-handle system on F embedded in V such that $V' = cl(V \setminus \beta)$ is a handlebody, [7].

Let $\beta_* = \beta_0 \cup \beta_1$, $\beta_i = \beta \times i$, and $V'_* = V'_0 \cup V'_1$, $V'_i = V' \times i$. Then $\partial V'_i = F_i(\beta_i)(i=0,1)$ and the union $W = h \cup V'_*$ is a handlebody with $\partial W = P(F)(h)(\beta_*)$. Let D_* be a transverse disk system of the 1-handle system β_*. It will be shown that there is a proper disk system D'_* in the handlebody W such that the pair $(D_* \times I, D'_* \times I)$ is an O2-handle pair system on $P(F)(h)(\beta_*)$. Let $h_0 = d \times [0,1]$ in $F \times [0,1]$ for a disk d in F, which is assumed to be a 1-handle on P(F) with the same attaching part as h. For any two spin loop bases (a,a′), (b,b′) of a trivial surface-knot T in S^4, there is an orientation- preserving diffeomorphism of (S^4,T) sending (a,a′) to (b,b′), [9], [11].

Further, for any O2-handle bases $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on T, there is an orientation preserving diffeomorphism of (S^4,T) sending $(D \times I, D' \times I)$ to $(E \times I, E' \times I)$, [4]. By using these results, there is an orientation-preserving diffeomorphism f of S^4 sending the handlebody W to $W' = h_0 \cup V'_*$ such that the restriction of f to V'_* is the identity map by moving $h \cup V'_1$ into $h_0 \cup V'_1$ and keeping V'_0 fixed. Then $\partial W' = P(F)(h_0)(\beta_*)$. By using the product $V' \times [0,1]$ in S^4, a proper disk system E'_* in W′ is constructed such that $(f(D_*) \times I, E'_* \times I)$ is an O2-handle pair system on $P(F)(h_0)(\beta_*)$. For the preimage $D'_* = f^{-1}(E'_*)$ of E'_* by f, the pair $(D_* \times I, D'_* \times I)$ is an O2-handle pair system on the surface-link $P(F)(h)(\beta_*)$ as desired. Note that the surface-knot G obtained from $P(F)(h)(\beta^*)$ by surgery along $(D_* \times I, D'_* \times I)$ is equivalent to the surface-knot obtained from $P(F)(h)(\beta_*)$ by surgery along the 2-handle system $D_* \times I$, [9]. Hence the surface-knot G is a trivial surface-knot and equivalent to P(F)(h), [4]. This completes the proof of Theorem 1.2.

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