Research Article
Open Access Journal of Physics & Mathematics
ISSN: 3068-7527
OPENPUB
GLOBAL PUBLISHERS
Received: 15-02-2025 Accepted: 19-02-2025 Published: 28-02-2025
Volume 1 | Issue 1

About a Solution to Beal's Conjecture

Dr. Chandramohanan MR*
Former Professor of Mathematics, Narayana guru College of Engineering Manjalummood, Tamil Nadu, India

***Corresponding author:** Chandramohanan MR, Former Professor of Mathematics, Narayanaguru College of Engineering, 41/2674 (1) Sreenilayam, Manacaud PO, Thiruvananthapuram, Kerala, India.

Citation: *Chandramohanan MR (2025) About a Solution to Beal's Conjecture. Open Access Journal of Physics & Mathematics, Research Article 1 (1): 01-10.*

Abstract

In this article, the solution of Beal's conjecture is presented. In the proof, the proof of Fermat's Last Theorem is also presented since in Beal's conjecture, the equation $x^a = y^b + z^c$ *becomes Fermat's diaphantine equation* $x^n = y^n + z^n$ *for* $a = b = c = n$*, a positive integer.*

Keywords: LCM for Lowest Common Multiple; $\mathrm{Max}(a,b,c)$ for the largest number among a, b, c; $\mathrm{Min}(a', b', c')$ for the least number among a', b', c'; GCD for Greatest Common Divisor; $O(n)$ and $M(n)$ have their usual meaning; FLT stands for Fermat's Last Theorem.

1. Introduction

The Fermat's Last Theorem [1,2,4] has its origin in 1637, Fermat noted on the margin of his personal copy of the Bachet edition of Diophantis-Arithametica that it is impossible to write a cube as a sum of two cubes, a fourth power as a sum of two fourth powers and in general, any power beyond the second, as a sum of two similar powers in positive integers. For this I discovered a truly wonderful proof, but the margin is too small to contain it. The above statement of Fermat is known as Fermat's Last Theorem (FLT). Despite efforts of many mathematicians and amateurs, it could not be proved for about 350 years. In June 1993, Andrew Wiles of Cambridge, UK announced a proof of FLT but it had some flaw [4,5]. Later Richard Taylor assisted him to correct the flaw. Together, Andrew Wiles and Richard Taylor published their proof of FLT for international scrutiny in May 1995 [4].

Inspired by the FLT, Andrew Beal, a banker from Texas, USA [4] proposed the following conjecture, known as Beal's conjecture: If

$$x^a = y^b + z^c \qquad (1)$$

where a, b, c are positive integers and may be different as well as greater than 2 and x, y, z are positive integers, have solutions, then x, y, z have a common factor greater than 2. The dissimilarity between FLT and Beal's equation is that, in FLT we consider values of x, y, z such that any two of them are coprimes whereas in Beal's equation, no two of them are coprimes but all the three have a common factor greater than 2.

Without loss of generality we may assume that $\mathbf{Max}(a,b,c) > 2, m = \mathbf{Min}(a',b',c') \geq 1$ where a', b', c' are found from $aa' = bb' = cc' = L = \mathrm{LCM}(a,b,c)$. There are two possibilities (i) a, b, c are not all equal (ii) $a = b = c = n$ for some positive integer.

2. Case 1. When a, b, c are not all equal

In this case, we assume there is an initial minimal solution (x_0, y_0, z_0) satisfying

$$x_0^a = y_0^b + z_0^c \qquad (2)$$

Suppose p is any odd prime or an even number from the set $\{2,4,8,16,32,\ldots\}$ if $m \geq 2$ or from the set $\{4,8,16,32,\ldots\}$ if $m = 1$. Now multiply (2) by p^L so that we have $\left(p^{a'}x_0\right)^a = \left(p^{b'}y_0\right)^b + \left(p^{c'}z_0\right)^c$ clearly p^m will be a common factor if the new values of $x = p^{a'}x_0, y = p^{b'}y_0, z = p^{c'}y_0$. But when $m = 1$ we restrict p to be either an odd prime or an even integer from the set $\{4,8,16,32,64,\ldots\}$ and when $m \geq 2$ we restrict p to be either an odd prime or an even number from the set $\{2,4,8,16,32,\ldots\}$

Examples:

$$3^2 = 2^3 + 1^3 \qquad \text{(i)}$$

$$5^3 = 11^2 + 2^2 \qquad \text{(ii)}$$

$$L = \mathrm{LCM}(2,3,3) = 6 \text{ and } \mathrm{LCM}\ (3,2,2) = 6$$

Clearly $\mathrm{Min}(a',b',c') = \mathrm{Min}(2,3,3)$ and $\mathrm{Min}(3,2,2) = 2$. Hence on multiplying (i) and (ii) by $p^6 = 3^6$ we note that

$$81^2 = 18^3 + 9^3 \qquad \text{(iii)}$$

and

$$45^3 = 297^2 + 54^2 \qquad \text{(iv)}$$

Clearly $p^2 = 9$ is a common factor of the new values of x, y, z and similarly multiplying (i) and (ii) by 2^6 the results are $24^2 = 8^3 + 4^3$ and $20^3 = 88^2 + 16^2$ so that $p^2 = 2^2 = 4$ is a common factor of the new values of x, y, z. On the other hand if we rewrite equation (i) as $3^2 = 2^3 + 1$ then minimum $a', b', c' = 2$ and on multiplying this by 2^6 the result is $24^2 = 8^3 + 64$ so that 4 and 8 are common factors of the new values of x, y, z. Hence it is true that $2^m = 2^2 = 4$ is a common factor. Thus Beal's conjecture is true according to restrictions stated, when a, b, c are not all equal. This completes the proof of Beal's conjecture in case 1.

In case 2, the proof will be complete by showing [3] that Fermat's Diophantine equation

$$x^n = y^n + z^n \qquad (3)$$

has (i) positive integral solutions for $n = 1$ and 2, but no positive integral solution when (ii) n is an odd positive integer and (iii) $n = 4$.

3. Case 2 (i) When $n = 1$ or 2

For $n = 1$ the equation (3) has a solution of the form $(p + q, p.q)$ where $p.q$ are coprimes. Hence multiplying this triplet by any positive integer greater than 2, the new values of x, y, z satisfy Beal's equations.

For $n = 2$ we rewrite (3) as

$$\left(\frac{x}{z}\right)^2 - \left(\frac{y}{z}\right)^2 = 1 \quad \text{or} \quad u^2 - v^2 = 1$$

where $u = \frac{x}{z}$, $v = \frac{y}{z}$ are positive rational numbers so that we may assume

$$u - v = \frac{p}{q} \qquad (4a)$$

and

$$u + v = \frac{q}{p} \qquad (4b)$$

where p, q are coprimes. Solving these last equations we have

$$u = \frac{q^2 + p^2}{2pq}, v = \frac{q^2 - p^2}{2pq}$$

so that

$$x = q^2 + p^2, y = q^2 - p^2, z = 2pq$$

will satisfy equation (3) for $n = 2$. By multiplying these values by any positive integer greater than 2, it is true that Beal's conjecture holds.

4. Case 2 (ii) Fermat's Last Theorem for an odd prime

From equation (3) we have

$$\left(\frac{x}{z}\right)^n - \left(\frac{y}{z}\right)^n = 1 \qquad (5)$$

or

$$u^n - v^n = 1 \qquad (6)$$

where we define $u = \frac{x}{z}, v = \frac{y}{z}$ such that $x - y = hp, z = hp$ where h is the HCF of $x - y$ and z and p, q are coprimes. Similarly we define $u' = \frac{x}{y}, v' = \frac{z}{y}$ such that $x - z = h'p', y = h'q'$ where h' is the HCF of $x - z$ and y and p', q' are coprimes. Factoring (6) we have

$$(u - v)(u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1}) = 1$$

Also

$$u - v = \frac{p}{q} \qquad (7)$$

$$\Rightarrow u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1} = \frac{q}{p} \qquad (8)$$

where $p < q$ since $x^n = y^n + z^n < (y + z)^n$ implies $x < y + z$. Similarly $n' - v' = \frac{p'}{q'}$ where $p' < q'$ and hence $u'^{n-1} + u'^{n-2}v' + \cdots + u'v'^{n-2} + v'^{n-1} = \frac{q'}{p'}$.

The points of intersection of (7) and (8) lie on (6) since the product of LHS of (7) and (8) and the RHS of (7) and (8) satisfy (6). If $\left(\frac{a}{b}, \frac{c}{d}\right)$ is any positive rational solution of (6) then by choosing $p = ad - bc, q = bd$, it is noted that this solution lies in the solution set of (7) and (8). Hence it is sufficient to solve (7) and (8) instead of (6). The equation of straight lines joining the origin to the points of intersection of (7) and (8) will be of the form

$$u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1} = \frac{q}{p}\left[(u - v)\frac{q}{p}\right]^{n-1} = \left(\frac{q}{p}\right)^n (u - v)^{n-1} \qquad (9)$$

Since (9) is a homogeneous equation in u, v of degree $n - 1$, it represents $(n - 1)$ straight lines through the origin of uv-plane, which may be real and/or imaginary. The slope $m = \frac{v}{u}$ of the straight lines contained in (9) satisfy

$$1 + m + \cdots + m^{n-1} = \left(\frac{q}{p}\right)^n (1-m)^{n-1}$$

That is

$$1 - m^n = \left(\frac{q}{p}\right)^n (1-m)^n \qquad (10)$$

Letting $m = \lambda + 1$ or

$$\lambda = m - 1 = \frac{v}{u} - 1 = \frac{y}{x} - 1 = \left(\frac{-hp}{x}\right)$$

we have

$$(\lambda+1)^n - 1 - \left(\frac{q}{p}\right)^n \lambda^n = 0$$

$$\text{i.e. } \binom{n}{1}\lambda + \binom{n}{2}\lambda^2 + \cdots + \binom{n}{n-1}\lambda^{n-1} + \left[1 - \frac{q^n}{p^n}\right]\lambda^n = 0$$

$$\text{i.e. } p^n\left[\binom{n}{1} + \binom{n}{2}\lambda + \cdots + \binom{n}{n-1}\lambda^{n-2}\right] - (q^n - p^n)\lambda^{n-1} = 0 \qquad (11)$$

$$\text{i.e. } xp\left[\binom{n}{1}x^{n-2} - \binom{n}{2}x^{n-3}(hp) + \cdots + \binom{n}{n-1}(hp)^{n-2}\right] = (q^n - p^n)h^{n-1}$$

$$\binom{n}{1}x^{n-1}(hp) - \binom{n}{2}x^{n-2}(hp)^2 + \cdots - \binom{n}{n-1}x(hp)^{n-1} = (hq)^n - (hp)^n \qquad (12)$$

Similarly

$$\binom{n}{1}x^{n-1}(h'p') - \binom{n}{2}x^{n-2}(h'p')^2 + \cdots - \binom{n}{n-1}x(h'p')^{n-1} = (h'q')^n - (h'p')^n \qquad (13)$$

It is clear that (12) and (13) are equivalent to (3) in disguise, but $nx(x - hp)(hp)$ is a factor of LHS of (12) and $nx(x - h'p')(h'p')$ is a factor of LHS of (13). Since $x^n = y^n + z^n$ implies $x \equiv y + z \bmod n$ by Fermat's little theorem, we have $h(q - p) \equiv 0 \equiv h'(q' - p') \bmod n$. These conditions give rise to four possibilities:

(i) $h \equiv 0 \bmod n$

(ii) $h' \equiv 0 \bmod n$

(iii) $q \equiv p \bmod n$ and

(iv) $q' \equiv p' \bmod n$.

First we shall show that (i) and (ii) are false. In order to prove the falsity of (ii) we show that (i) $h' \equiv 0 \equiv h \bmod$ and (ii) $h' \equiv 0 \equiv (q - p) \bmod n$ are false. Then by using the principle of symbolic logic in the form, if *A, B, C* are logical statement satisfying the conditions (i) $T(B \vee C) = 1$ (ii) $T(A \wedge B) = 0 = T(A \wedge C)$ then $T(A) = 0$ or *A* is false. This follows from the fact that $T[A \wedge (B \vee C)] = T[(A \wedge B) \vee (A \wedge C)] \leq T(A \wedge B) + T(A \wedge C) = 0 + 0$. So that $T[A \wedge (B \vee C)] = 0$ therefore $T(A) = 0$ where the truth value function *T* is non-negative. It is possible to define $A \equiv (h' \equiv 0 \bmod n)$, $B \equiv (h \equiv 0 \bmod n)$ and $C \equiv (q \equiv p \bmod n)$ so that $B \vee C \equiv (h(q - p) \equiv 0 \bmod n)$ which has truth value 1, by the assumption that there exists a positive integral solution of equation (1).

According to the possibility (i) $h' \equiv 0 \equiv h \bmod n$ we have $x - y \equiv 0 \equiv z \bmod n$ and $x - z \equiv 0 \equiv y \bmod n$ implying that $x \equiv y \equiv z \equiv 0 \bmod n$ so that *n* is a common factor of *x*, *y*, *z* which contradicts the assumption that GCD $(x, y, z) = 1$. This contradiction proves that the possibility (i) $h' \equiv 0 \equiv h \bmod n$ is false. According to the possibility (ii) $h' \equiv 0 \equiv (q - p) \bmod n$ we have $x - z \equiv 0 \equiv y \bmod n$ and $q = p + kn$ for some positive integer *k* so that $h'p' = M(n)$ and $h'q' = M(n)$ and hence the RHS of (13) is $M(n^n)$ and on LHS we have the first term $= M(n^2)$ unless $x \equiv 0 \bmod n$ but the remaining terms are $M(n^3)$ at least, due to the presence of $(h'p')$.

The LHS of (13) consists of terms of order $M(n^2), M(n^3)$ etc. due to the presence of $h'p'$ so that all terms except the first term are divisible by n^3 and RHS is also divisible by n^3, since RHS $= (hq)^n - (hp)^n = M(n^n)$ is divisible by n^3, but the first term of LHS is not divisible by n^3 unless $x = M(n)$. This disparity implies that we must have $x = M(n)$ in addition to $y = h'q' = M(n) \Rightarrow x, y, z$ are $M(n)$.

This contradiction proves that the possibility (ii) $h' \equiv 0 \equiv (q - p) \bmod n$ is false. Since both possibilities (i) and (ii) are false, it follows that $h' \equiv 0 \bmod n$ is false, by the principle of symbolic logic stated earlier. Similarly $h \equiv 0 \bmod n$ is also false. $\therefore h' \not\equiv 0 \not\equiv h \bmod n$. Now we are left with the possibilities (iii) $q \equiv p \bmod n$ and (iv) $q' \equiv p' \bmod n$ or equivalently.

$$q = p + Kn \tag{14}$$

and

$$q' = p' + K'n \tag{15}$$

for some positive integers K and K'. It will be shown that (14) and (15) are false statements. (14) implies

$$q^n - p^n = (p + Kn)^n - p^n = \sum_{r=0}^{n-1} \binom{n}{r} p^r (Kn)^{(n-r)}$$

is divisible by n^2 at least, since n divides $\binom{n}{r}$ for $0 < r < n$ and $n - r > 2$ for $r = 0. \Rightarrow (hq)^n - (hp)^n$ is divisible by n^2.

Next, we claim that $p \equiv 0 \bmod n$ and $q \equiv 0 \bmod n$ are false for the following reasons (i) $p \equiv 0 \equiv q \bmod n$ implies that n is a common factor of p and q which are coprimes (ii) $p \equiv 0 \not\equiv q \bmod n$ implies that $x \equiv y \bmod n$ but $z \not\equiv 0 \bmod n$. This is absurd since $x \equiv y \bmod n \Rightarrow z \equiv 0 \bmod n$. (iii) $q \equiv 0 \not\equiv p \bmod n$ implies that $z \equiv 0 \bmod n$ but $x \not\equiv y \bmod n$ which is false since $z \equiv 0 \bmod n$ implies $x \equiv y \bmod n$. Hence p, q are not divisible by n. Similarly p', q' are not divisible by n and so also are $y = h'q'$ and $z = hq$.

Rewriting equation (12) we have

$$\left[\binom{n}{1} x^{n-1}(hp) - \binom{n}{n-1} x(hp)^{n-1}\right] - \left[\binom{n}{2} x^{n-2}(hp)^2 - \binom{n}{n-2} x^2 (hp)^{n-2}\right] + \cdots = (hq)^n - (hp)^n \tag{16}$$

On the LHS of equation (16) the first bracket is divisible by $\binom{n}{1}$, so that the first bracket is of order $M(n)$ and remaining brackets on LHS are of order at least $O(n^2)$ but all these brackets turn out to be $M(n^2)$ by choosing $x(hp) = M(n)$ and the RHS is divisible by n^2. This disparity demands us to let $x(hp) = M(n) \Rightarrow x = M(n)$. Since $h \neq M(n)$ and $p \neq M(n)$.

If $x = M(n)$ then $y + z = M(n) = \bar{K}n$ for some positive integer $\bar{K}$.

$$\therefore (y + z)^n - (y^n + z^n) = \sum_{r=1}^{n-1} \binom{n}{r} y^r z^{n-r} \Rightarrow$$

$\therefore M(n^n) = yzO(n^t)$ where $t = \frac{n-1}{2}$ so that $yz = O(n^{t+1}) = O(n)$ at least $= M(n)$ at least since yz is a positive integer. This is also clear from the fact that n divides $\binom{n}{r}$ for $0 < r < n$ but $y^{r-1} z^{n-r-1}$ of degree

$n-2$ cannot give rise to a factor of order $M(n^{n-1})$ or more.

$$\text{since } y^n + z^n = x^n = M(n^n) \text{ and } (y+z)^n = M(n^n)$$

$$\therefore yz = M(n)$$

which contradicts the earlier assertions $h \not\equiv 0 \not\equiv h' \bmod n$, $q \not\equiv 0 \not\equiv q' \bmod n$ and $y = h'q' \neq M(n)$, $z = hq \neq M(n)$. This implies equations (14) and (15) are false. Hence the possibilities (iii) and (iv) stated after equation (13) are false. Thus all the four possibilities (i) to (iv) stated after equation (13) are false. Hence $q-p$ and $q'-p'$ are not divisible by $n \Rightarrow q^n - p^n = q - p + M(n)$ and $q'^n - p'^n = (q'-p') + M(n)$ are not divisible by n.

Hence the polynomial in λ of equation (11) satisfies Eisenstein's criterion [1,3] for irreducibility over $\mathbb{Q}$ since $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$, are divisible by n but $n \nmid p^n, n \nmid (q^n - p^n)$ and $n^2 \nmid \binom{n}{1} p^n$. Therefore the roots of (11) are irrational. Hence λ and $\mathrm{m} = \lambda + 1$ can have only irrational values $\Rightarrow$ Equations (7) and (8) and hence (6) cannot have a positive rational solution. This proves that equation (3) has no positive integral solution. Hence FLT is true for any odd prime n. This is also clear from the fact that we have shown the falsity of possibilities (i) to (iv) stated after equation (13).

5. Case 2 (iii) Fermat's Last Theorem for $n = 4$

By letting $n = 4$ in equations (5) to (10), the equation (10) becomes[5]

$$1 - m^4 = \frac{q^4}{p^4}(1-m)^4$$

Letting $m = \lambda + 1$ or $\lambda = m - 1 = \left(\frac{-hp}{x}\right)$ we have

$$(\lambda+1)^4 - 1 + \frac{q^4}{p^4}\lambda^4 = 0 \Rightarrow \left(1 + \frac{q^4}{p^4}\right)\lambda^4 + 4\lambda^2 + 6\lambda^2 + 4\lambda = 0$$

$$(p^4 + q^4)\lambda^3 + 2p^4(2\lambda^2 + 5\lambda + 2) = 0 \qquad (17)$$

$$(p^4+q^4)\left(\frac{-h^3p^3}{x^3}\right)+2p^4\left[2\left(\frac{h^2p^2}{x^2}\right)-3\left(\frac{hp}{x}\right)+2\right]$$

$$\therefore \frac{(p^4+q^4)h^3}{p}=2x[2h^2p^2-3hpx+2x^2] \qquad (18)$$

Similarly

$$\frac{(p'^4+q'^4)}{p'}h'^3=2x[2h'^2p'^2-3h'p'x+2x^2] \qquad (19)$$

From $x^4=y^4+z^4$ we have $x\equiv y+z \bmod 2$ so that $x-y\equiv z \bmod 2$ and $x-z\equiv y \bmod 2$ or $h(q-p)\equiv 0\equiv h'(q'-p') \bmod 2$. In order to prove the falsity of $h'\equiv 0 \bmod 2$, we consider the possibilities (i) $h'\equiv 0\equiv h \bmod 2$ and (ii) $h'\equiv 0\equiv (q-p) \bmod 2$ and prove that these conditions are invalid as in Section '**Fermat's Last Theorem for an odd prime**'.

In the former possibility, we have $x-z\equiv 0\equiv y \bmod 2$ and $x-y\equiv 0\equiv z \bmod 2$ implying that *x, y, z* are even positive integers. This contradiction proves that possibilities (i) is invalid. In the possibility (ii) we have $h'\equiv 0 \bmod 2$ and $q\equiv p \bmod 2$ so that h' is even and *p*, *q* are odd positive integers, since they are coprimes. Also $y=h'q'$ is even and hence *x*, *z* are odd $h'p'$ is also even. If $y=h'q'$ is even then we may write $y=2y_1, x-z=2k_1y_1, x+z=2k_2y_1$ where k_1 and k_2 are positive integers so that $x^2+z^2=2(k_1^2+k_2^2)y_1^2$ and $y^4=x^4-z^4$ implies $16y_1^4=2k_1y_1.2k_2y_1.2(k_1^2+k_2^2)y_1^2=8k_1k_2(k_1^2+k_2^2)y_1^4$. That is $k_1k_2(k_1^2+k_2^2)=2$. This has solution $k_1=k_2=1$ only leading to $z=0$ and $x=y$, the trivial solution for FLT. This proves that *y* cannot be even. $\Rightarrow h'$ and q' must be odd. If p' is even, then $h'p'=x-z$ is even $\Rightarrow y=x-z+M(2)=M(2)$ which is already ruled out above. Hence h', p', q' are all odd positive integers. Therefore the statement $h'\equiv 0 \bmod 2$ and $q\equiv p \bmod 2$ is false. Similarly h, p, q are all odd and statement $h\equiv 0 \bmod 2$ and $q'\equiv p' \bmod 2$ is false.

Hence $y=hq, z=h'q'$ are odd integers and hence *x* must be an even positive integer. We shall show that this statement is invalid. Suppose $x^4=y^4+z^4$ where x, y, z are positive integers such that their GCD = 1. Letting $X=x^2, Y=y^2, Z=z^2$ we have $X^2=Y^2+Z^2$ so that (X, Y, Z) from a Pythagorean triple with solution

$$X = Q^2 + P^2, Y = Q^2 - P^2, Z = 2QP$$

where P, Q are coprimes (with $Q > P$) in which Y and Z can be exchanged due to symmetry. Clearly Z/Y is even so that X must be an odd integer.

$\therefore x^2$ and hence x must be an odd integer. Hence the requirement that x is even and y, z are odd, cannot be satisfied. Also y/z is even implies q/q' is even contradicting the earlier assertion that q and q' are odd. These contradictions prove that $x^4 = y^4 + z^4$ has no positive integral solution i.e. FLT is true for $n = 4$. This completes the proof of Beal's conjecture.

References

1. Archbold J (1972) Algebra, ELBS and Pitman, London.
2. Burton DM (2007) The History of Mathematics, an Introduction, McGrew Hill, New York.
3. Escoffier JP (2001) Galois Groups. Galois Theory 119-148.
4. Hassannaba B (2015) Incredible Mathematical Stories, Prism Books Pvt. Ltd., Bangaluru.
5. Chandramohanan MR (2024) About Fermat's Last Theorem, Journal of Mathematical and Computer Applications 6.